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Normal Approximations with Malliavin Calculus

From Stein's Method to Universality

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To Lili, Juliette and Delphine.
To Emma Elīza and Ieva.

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Preface

This is a text about *probabilistic approximations*, which are mathematical statements providing estimates of the distance between the laws of two random objects. As the title suggests, we will be mainly interested in approximations involving one or more *normal* (equivalently called *Gaussian*) random elements. Normal approximations are naturally connected with central limit theorems (CLTs), i.e. convergence results displaying a Gaussian limit, and are one of the leading themes of the whole theory of probability.

The main thread of our text concerns the normal approximations, as well as the corresponding CLTs, associated with random variables that are functionals of a given Gaussian field, such as a (fractional) Brownian motion on the real line. In particular, a pivotal role will be played by the elements of the so-called *Gaussian Wiener chaos*. The concept of Wiener chaos generalizes to an infinite-dimensional setting the properties of the Hermite polynomials (which are the orthogonal polynomials associated with the one-dimensional Gaussian distribution), and is now a crucial object in several branches of theoretical and applied Gaussian analysis.

The cornerstone of our book is the combination of two probabilistic techniques, namely the *Malliavin calculus of variations* and *Stein's method* for probabilistic approximations.

The Malliavin calculus of variations is an infinite-dimensional differential calculus, whose operators act on functionals of general Gaussian processes. Initiated by Paul Malliavin (starting from the seminal paper [69], which focused on a probabilistic proof of Hörmander's 'sum of squares' theorem), this theory is based on a powerful use of infinite-dimensional integration by parts formulae. Although originally exploited for studying the regularity of the laws of Wiener functionals (such as the solutions of stochastic differential equations), the scope of its actual applications, ranging from density estimates to concentration inequalities, and from anticipative stochastic calculus to the computations of 'Greeks' in mathematical finance, continues to grow. For a classic presentation of this subject, the reader can consult the three texts by Malliavin [70], Nualart [98] and Janson [57]. Our book is the first monograph providing a self-contained introduction to Malliavin calculus from the specific standpoint of limit theorems and probabilistic approximations.

Stein's method can be roughly described as a collection of probabilistic techniques for assessing the distance between probability distributions by means of differential operators. This approach was originally developed by Charles Stein in the landmark paper [135], and then further refined in the monograph [136]. In recent years, Stein's method has become one of the most popular and powerful tools for computing explicit bounds in probabilistic limit theorems, with applications to fields as diverse as random matrices, random graphs, probability on groups and spin glasses (to name but a few). The treatise [22], by Chen, Goldstein and Shao, provides an exhaustive discussion of the theoretical foundations of Stein's method for normal approximations, as well as an overview of its many ramifications and applications (see also the two surveys by Chen and Shao [23] and Reinert [117]).

We shall show that the integration by parts formulae of Malliavin calculus can be fruitfully combined with the differential operators arising in Stein's method. This interaction will be exploited to produce a set of flexible and far-reaching tools, allowing general CLTs (as well as explicit rates of convergence) to be deduced for sequences of functionals of Gaussian fields. It should be noted that the theory developed in this book virtually replaces every technique previously used to establish CLTs for Gaussian-subordinated random variables, e.g. those based on moment/cumulant computations (see, for example, Peccati and Taqqu [110]).

As discussed at length in the text, the theoretical backbone of the present monograph originates from the content of five papers.

- Nualart and Peccati [101] give an exhaustive (and striking) characterization of CLTs inside a fixed Wiener chaos. This result, which we will later denote as the 'fourth-moment theorem', yields a drastic simplification of the classic *method of moments and cumulants*, and is one of the main topics discussed in the book.
- Peccati and Tudor [111] provide multidimensional extensions of the findings of [101]. In view of the *Wiener–Itô chaotic representation property* (see Chapter 2), the findings of [111] pave the way for CLTs involving general functionals of Gaussian fields (not necessarily living inside a fixed Wiener chaos).
- The paper by Nualart and Ortiz-Latorre [100] contains a crucial methodological breakthrough, linking CLTs on Wiener chaos to the

asymptotic behavior of Malliavin operators. In particular, a prominent role is played by the norms of the *Malliavin derivatives* of multiple Wiener–Itô integrals.

- Nourdin and Peccati [88] establish the above-mentioned connection between Malliavin calculus and Stein's method, thus providing substantial refinements of the findings of [100, 101, 111]. Along similar lines, the multivariate case is dealt with by Nourdin, Peccati and Réveillac [95].
- Nourdin, Peccati and Reinert [94] link the above results to the so-called *universality phenomenon*, according to which the asymptotic behavior of (correctly rescaled) large random systems does not depend on the distribution of their components. Universality results, also known as 'invariance principles', are almost ubiquitous in probability: distinguished examples are the classic central limit theorem and the circular and semicircular laws in random matrix theory. See Chapter 11 for further discussions.

The above-mentioned references have been the starting point of many developments and applications. These include density estimates, concentration inequalities, Berry–Esseen bounds for power variations of Gaussian-subordinated processes, normalization of Brownian local times, random polymers, random matrices, parametric estimation in fractional models and the study of polyspectra associated with stationary fields on homogeneous spaces. Several of these extensions and applications are explicitly described in our book. See the webpage

`http://www.iecn.u-nancy.fr/~nourdin/steinmalliavin.htm`

for a constantly updated reference list. See the monographs [74] and [110], respectively, for further applications to random fields on the sphere (motivated by cosmological data analysis), and for a discussion of the combinatorial structures associated with the Gaussian Wiener chaos.

The book is addressed to researchers and graduate students in probability and mathematical statistics, wishing to acquire a thorough knowledge of modern Gaussian analysis as used to develop asymptotic techniques related to normal approximations.

With very few exceptions (where precise references are given), every result stated in the book is proved (sometimes through a detailed exercise), and even the most basic elements of Malliavin calculus and Stein's method are motivated, defined and studied from scratch. Several proofs are new, and each chapter contains a set of exercises (with some hints!), as well as a number of bibliographic comments. Due to these features, the text is more or less self-contained, although our ideal reader should have attended a basic course in modern probability (corresponding, for example, to the books by Billingsley [13] and Chung [25]), and also have some knowledge of functional analysis (covering, for example, the content of chapters 5–6 in Dudley's book [32]). Some facts and definitions concerning operator theory are used – we find that a very readable reference in this respect is [47], by Hirsch and Lacombe.

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