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978-1-107-01777-1 - Normal Approximations with Malliavin Calculus: From Stein's Method to Universality
Ivan Nourdin and Giovanni Peccati
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Introduction

Let $F = (F_n)_{n\geq 1}$ be a sequence of random variables, and assume that F satisfies a *central limit theorem* (CLT), that is, there exists a (non-zero) Gaussian random variable Z such that, as $n \to \infty$,

$$P(F_n \leq z) \to P(Z \leq z), \quad \text{for every } z \in \mathbb{R}$$

(here $(\Omega, \mathcal{F}, P)$ denotes the underlying probability space). A natural question is therefore the following: *for a fixed n, can we assess the distance between the laws of F_n and Z?* In other words, *is it possible to quantify the error one makes when replacing F_n with Z?* Answering questions of this type customarily requires us to produce uniform upper bounds of the type

$$\sup_{h\in\mathscr{H}} |E[h(F_n)] - E[h(Z)]| \leq \phi(n), \quad n \geq 1,$$

where $\mathscr{H}$ is a rich enough class of test functions, E denotes mathematical expectation, and $(\phi(n))_{n\geq 1}$ is a positive numerical sequence (sometimes called a *rate of convergence*) such that $\phi(n) \to 0$. Finding explicit rates of convergence can have an enormous impact on applications. For instance, if F_n is some statistical estimator with unknown distribution, then a small value of $\phi(n)$ implies that a Gaussian likelihood may be appropriate; if F describes the evolution of some random system exhibiting asymptotically Gaussian fluctuations, then a rate $\phi(n)$ rapidly converging to zero implies that one can safely consider such fluctuations to be Gaussian, up to some negligible error.

The aim of this monograph is to build an exhaustive theory, allowing the above questions to be answered whenever the sequence F is composed of sufficiently regular functionals of a (possibly infinite-dimensional) Gaussian field. As made clear by the title, our main tools will be the *Malliavin calculus of variations* and *Stein's method for normal approximations*. Both topics will be developed from first principles.

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The book is organized as follows:

- **Chapter 1** deals with Malliavin operators in the special case where the underlying Gaussian space is one-dimensional. This chapter is meant as a 'smooth' introduction to Malliavin calculus, as well as to some more advanced topics discussed later in the book, such as variance expansions and second-order Poincaré inequalities. Several useful computations concerning one-dimensional Gaussian distributions and Hermite polynomials are also carefully developed.
- **Chapter 2** contains all the definitions and results on Malliavin calculus that are needed throughout the text. Specific attention is devoted to the derivative and divergence operators, and to the properties of the so-called Ornstein–Uhlenbeck semigroup. The notions of Wiener chaos, Wiener–Itô multiple integrals and chaotic decompositions are also introduced from scratch.
- **Chapter 3** introduces Stein's method for normal approximations in the one-dimensional case. General Stein's equations are studied in detail, and Stein-type bounds are obtained for the total variation, Kolmogorov and Wasserstein distances.
- **Chapter 4** discusses multidimensional Gaussian approximations. The main proofs in this chapter are based on the use of Malliavin operators.
- **Chapter 5** is arguably the most important of the book. Here we show how to explicitly combine Stein's method with Malliavin calculus. One of the main achievements is a complete characterization of CLTs on a fixed Wiener chaos, in terms of 'fourth-moment conditions'. Several examples are discussed in detail.
- **Chapter 6** extends the findings of Chapter 5 to the multidimensional case. In particular, the results of this chapter yield some useful characterizations of CLTs for vectors of chaotic random variables.
- **Chapter 7** contains a detailed application to the so-called *Breuer–Major CLTs*. These convergence results are one of the staples of asymptotic Gaussian analysis. They typically involve sequences of the form $F_n = n^{-1/2}\sum_{i=1}^{n} f(X_i)$, $n \geq 1$, where $(X_i)_{i\geq 1}$ is a stationary Gaussian sequence (for instance, given by the increments of a fractional Brownian motion) and f is some deterministic mapping. This framework is both very natural and very challenging, and is perfectly tailored to demonstrate the power of the techniques developed in the preceding chapters.
- In **Chapter 8** we provide some applications of Malliavin calculus to the recursive computations of cumulants of (possibly vector-valued) random elements. The results of this section may be seen as a simpler alternative

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to the familiar moments/cumulants computations based on graphs and diagrams (see [110]).

- **Chapter 9** deals with the delicate issue of optimality of convergence rates. Some connections with Edgeworth expansions are also discussed.
- **Chapter 10** deals with some explicit formulae for the density of the laws of functionals of Gaussian fields. This chapter, which is mainly based on [96], provides expressions for densities that differ from those usually obtained via Malliavin calculus (see, for example, [98, chapter 1]).
- **Chapter 11** establishes an explicit connection between the previous material and the so-called 'universality phenomenon'. The results of this chapter are tightly connected with a truly remarkable paper by Mossel, O'Donnel and Oleszkiewicz (see [79]), providing an extension of the Lindeberg principle to the framework of polynomial functionals of sequences of independent random variables.
- The book concludes with five appendices. **Appendix A** deals with Gaussian random variables, cumulants and Edgeworth expansions. **Appendix B** focuses on Hilbert spaces and contractions. **Appendix C** collects some useful results about distances between probability measures. **Appendix D** is an introduction to fractional Brownian motion. Finally, **Appendix E** discusses some miscellaneous results from functional analysis.

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1

Malliavin operators in the one-dimensional case

As anticipated in the Introduction, in order to develop the main tools for the normal approximations of the laws of random variables, we need to define and exploit a modicum of *Malliavin-type operators* – such as the *derivative*, *divergence* and *Ornstein–Uhlenbeck* operators. These objects act on random elements that are functionals of some Gaussian field, and will be fully described in Chapter 2. The aim of this chapter is to introduce the reader into the realm of Malliavin operators, by focusing on their one-dimensional counterparts. In particular, in what follows we are going to define derivative, divergence and Ornstein–Uhlenbeck operators acting on random variables of the type $F = f(N)$, where f is a deterministic function and $N \sim \mathscr{N}(0, 1)$ has a standard Gaussian distribution. As we shall see below, one-dimensional Malliavin operators basically coincide with familiar objects of functional analysis. As such, one can describe their properties without any major technical difficulties. Many computations detailed below are further applied in Chapter 3, where we provide a thorough discussion of Stein's method for one-dimensional normal approximations.

For the rest of this chapter, every random object is defined on an appropriate probability space $(\Omega, \mathscr{F}, P)$. The symbols 'E' and 'Var' denote, respectively, the expectation and the variance associated with P.

1.1 Derivative operators

Let us consider the probability space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \gamma)$, where γ stands for the standard Gaussian probability measure, that is,

$$\gamma(A) = \frac{1}{\sqrt{2\pi}} \int_A e^{-x^2/2} dx,$$

for every Borel set A. A random variable N with distribution γ is called *standard Gaussian*; equivalently, we write $N \sim \mathscr{N}(0, 1)$. We start with a simple (but crucial) statement.

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Lemma 1.1.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function such that $f' \in L^1(\gamma)$. Then $x \mapsto xf(x) \in L^1(\gamma)$ and*

$$\int_{\mathbb{R}} xf(x)d\gamma(x) = \int_{\mathbb{R}} f'(x)d\gamma(x). \tag{1.1.1}$$

Proof Since $\int_{\mathbb{R}} |x|d\gamma(x) < \infty$, we can assume that $f(0) = 0$ without loss of generality. We first prove that the mapping $x \mapsto xf(x)$ is in $L^1(\gamma)$. Indeed,

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|\,|x|d\gamma(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\int_0^x f'(y)dy\right| |x|\,e^{-x^2/2}dx\\ &\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\left(\int_x^0 |f'(y)|dy\right)(-x)e^{-x^2/2}dx\\ &\quad+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\left(\int_0^x |f'(y)|dy\right)xe^{-x^2/2}dx\\ &=\int_{-\infty}^{\infty}|f'(y)|d\gamma(y) < \infty,\end{aligned}$$

where the last equality follows from a standard application of the Fubini theorem. To show relation (1.1.1), one can apply once again the Fubini theorem and infer that

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)xd\gamma(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\left(\int_x^0 f'(y)dy\right)(-x)e^{-x^2/2}dx\\ &\quad+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\left(\int_0^x f'(y)dy\right)xe^{-x^2/2}dx=\int_{-\infty}^{\infty} f'(y)d\gamma(y).\end{aligned}$$

□

Remark 1.1.2 Due to the fact that the assumptions in Lemma 1.1.1 are minimal, we proved relation (1.1.1) by using a Fubini argument instead of a (slightly more natural) integration by parts. Observe that one cannot remove the 'absolutely continuous' assumption on f. For instance, if $f = \mathbf{1}_{[0,\infty)}$, then $\int_{-\infty}^{\infty} xf(x)d\gamma(x) = \frac{1}{\sqrt{2\pi}}$, whereas $\int_{-\infty}^{\infty} f'(x)d\gamma(x) = 0$.

We record a useful consequence of Lemma 1.1.1, consisting in a characterization of the moments of γ, which we denote by

$$m_n(\gamma) = \int_{\mathbb{R}} x^n d\gamma(x), \quad n \ge 0. \tag{1.1.2}$$

Corollary 1.1.3 *The sequence $(m_n(\gamma))_{n\ge 0}$ satisfies the induction relation*

$$m_{n+1}(\gamma) = n \times m_{n-1}(\gamma), \quad n \ge 0. \tag{1.1.3}$$

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In particular, one has $m_n(\gamma) = 0$ if n is odd, and

$$m_n(\gamma) = n!/(2^{n/2}(n/2)!) = (n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-1) \quad \text{if } n \text{ is even.}$$

Proof To obtain the induction relation (1.1.3), just apply (1.1.1) to the function $f(x) = x^n$, $n \geq 0$. The explicit value of $m_n(\gamma)$ is again computed by an induction argument. □

In what follows, we will denote by $\mathcal{S}$ the set of $\mathcal{C}^\infty$-functions $f : \mathbb{R} \to \mathbb{R}$ such that f and all its derivatives have at most polynomial growth. We call any element of $\mathcal{S}$ a *smooth* function.

Remark 1.1.4 The relevance of smooth functions is explained by the fact that the operators introduced below are all defined on domains that can be obtained as the closure of $\mathcal{S}$ with respect to an appropriate norm. We will see in the next chapter that an analogous role is played by the collection of the *smooth functionals* of a general Gaussian field. In the one-dimensional case, the reason for the success of this 'approximation procedure' is nested in the following statement.

Proposition 1.1.5 *The monomials $\{x^n : n = 0, 1, 2, \ldots\}$ generate a dense subspace of $L^q(\gamma)$ for every $q \in [1, \infty)$. In particular, for any $q \in [1, \infty)$ the space $\mathcal{S}$ is a dense subset of $L^q(\gamma)$.*

Proof Elementary Hahn–Banach theory (see Proposition E.1.3) implies that it is sufficient to show that, for every $\eta \in (1, \infty]$, if $g \in L^\eta(\gamma)$ is such that $\int_{\mathbb{R}} g(x)x^k d\gamma(x) = 0$ for every integer $k \geq 0$, then $g = 0$ almost everywhere. So, let $g \in L^\eta(\gamma)$ satisfy $\int_{\mathbb{R}} g(x)x^k d\gamma(x) = 0$ for every $k \geq 0$, and fix $t \in \mathbb{R}$. We have, for all $x \in \mathbb{R}$,

$$\left| g(x)e^{-\frac{x^2}{2}} \sum_{k=0}^{n} \frac{(itx)^k}{k!} \right| \leq |g(x)|e^{|tx|-\frac{x^2}{2}},$$

so that, by dominated convergence,[1] we have

$$\int_{\mathbb{R}} g(x)e^{itx} d\gamma(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(it)^k}{k!} \int_{\mathbb{R}} g(x)x^k d\gamma(x) = 0.$$

[1] Indeed, by the Hölder inequality and by using the convention $\frac{\infty-1}{\infty} = 1$ to deal with the case $\eta = \infty$, one has that

$$\int_{-\infty}^{\infty} |g(x)|e^{|tx|-\frac{x^2}{2}} dx = \sqrt{2\pi} \int_{-\infty}^{\infty} |g(x)|e^{|tx|} d\gamma(x) \leq \sqrt{2\pi} \|g\|_{L^\eta(\gamma)} \|e^{|t\cdot|}\|_{L^{(\eta-1)/\eta}(\gamma)} < \infty.$$

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We have therefore proved that $\int_{\mathbb{R}} g(x)\exp(itx)d\gamma(x) = 0$ for every $t \in \mathbb{R}$, from which it follows immediately (by injectivity of the Fourier transform) that $g = 0$ almost everywhere. □

Fix $f \in \mathcal{S}$; for every $p = 1, 2, \ldots$, we write $f^{(p)}$ or, equivalently, $D^p f$ to indicate the pth derivative of f. Note that the mapping $f \mapsto D^p f$ is an operator from $\mathcal{S}$ into itself. We now prove that this operator is closable.

Lemma 1.1.6 *The operator $D^p : \mathcal{S} \subset L^q(\gamma) \to L^q(\gamma)$ is closable for every $q \in [1, \infty)$ and every integer $p \geq 1$.*

Proof We only consider the case $q > 1$; due to the duality $L^1(\gamma)/L^\infty(\gamma)$, the case $q = 1$ requires some specific argument and is left to the reader. Let (f_n) be a sequence of $\mathcal{S}$ such that: (i) f_n converges to zero in $L^q(\gamma)$; (ii) $f_n^{(p)}$ converge to some η in $L^q(\gamma)$. We have to prove that η is equal to zero. Let $g \in \mathcal{S}$, and define $\delta^p g \in \mathcal{S}$ iteratively by $\delta^r g = \delta^1 \delta^{r-1} g$, $r = 2, \ldots, p$, where $\delta^1 g(x) = \delta g(x) = xg(x) - g'(x)$ (note that this notation is consistent with the content of Section 1.2, where the operator δ will be fully characterized). We have, using Lemma 1.1.1 several times,

$$\begin{aligned}\int_{\mathbb{R}} \eta(x)g(x)d\gamma(x) &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n^{(p)}(x)g(x)d\gamma(x)\\ &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n^{(p-1)}(x)\big(xg(x) - g'(x)\big)d\gamma(x)\\ &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n^{(p-1)}(x)\delta g(x)d\gamma(x)\\ &= \ldots\\ &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\delta^p g(x)d\gamma(x).\end{aligned}$$

Hence, since $f_n \to 0$ in $L^q(\gamma)$ and $\delta^p g$ belongs to $\mathcal{S} \subset L^{\frac{q}{q-1}}(\gamma)$, we deduce, by applying the Hölder inequality, that $\int_{\mathbb{R}} \eta(x)g(x)d\gamma(x) = 0$. Since it is true for any $g \in \mathcal{S}$, we deduce from Proposition 1.1.5 that $\eta = 0$ almost everywhere, and the proof of the lemma is complete. □

Fix $q \in [1, \infty)$ and an integer $p \geq 1$. We set $\mathbb{D}^{p,q}$ to be the closure of $\mathcal{S}$ with respect to the norm

$$\|f\|_{\mathbb{D}^{p,q}} = \left(\int_{\mathbb{R}} |f(x)|^q d\gamma(x) + \int_{\mathbb{R}} |f'(x)|^q d\gamma(x) + \ldots \right.$$
$$\left. + \int_{\mathbb{R}} |f^{(p)}(x)|^q d\gamma(x)\right)^{1/q}.$$

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In other words, a function f is an element of $\mathbb{D}^{p,q}$ if and only if there exists a sequence $(f_n)_{n\geq 1} \subset \mathcal{S}$ such that (as $n \to \infty$): (i) f_n converges to f in $L^q(\gamma)$; and (ii) for every $j = 1, \ldots, p$, $f_n^{(j)}$ is a Cauchy sequence in $L^q(\gamma)$. For such an f, one defines

$$f^{(j)} = D^j f = \lim_{n\to\infty} D^j f_n = \lim_{n\to\infty} f_n^{(j)}, \tag{1.1.4}$$

where $j = 1, \ldots, p$, and the limit is in the sense of $L^q(\gamma)$. Observe that

$$\mathbb{D}^{p,q+\epsilon} \subset \mathbb{D}^{p+m,q}, \quad \forall m \geq 0,\ \forall \epsilon \geq 0. \tag{1.1.5}$$

We write $\mathbb{D}^{\infty,q} = \bigcap_{p\geq 1} \mathbb{D}^{p,q}$.

Remark 1.1.7 Equivalently, $\mathbb{D}^{p,q}$ is the Banach space of all functions in $L^q(\gamma)$ whose derivatives up to the order p in the sense of distributions also belong to $L^q(\gamma)$ – see, for example, Meyers and Serrin [78].

Definition 1.1.8 For $p = 1, 2, \ldots$, the mapping

$$D^p : \mathbb{D}^{p,q} \to L^q(\gamma) : f \mapsto D^p f, \tag{1.1.6}$$

as defined in (1.1.4), is called the pth **derivative operator** (associated with the $L^q(\gamma)$ norm). Note that, for every $q \neq q'$, the operators $D^p : \mathbb{D}^{p,q} \to L^q(\gamma)$ and $D^p : \mathbb{D}^{p,q'} \to L^{q'}(\gamma)$ coincide when acting on the intersection $\mathbb{D}^{p,q} \cap \mathbb{D}^{p,q'}$. When $p = 1$, we will often write D instead of D^1.

Since $L^2(\gamma)$ is a Hilbert space, the case $q = 2$ is very important,. In the next section, we characterize the adjoint of the operator $D^p : \mathbb{D}^{p,2} \to L^2(\gamma)$.

1.2 Divergences

Definition 1.2.1 We denote by $\mathrm{Dom}\,\delta^p$ the subset of $L^2(\gamma)$ composed of those functions g such that there exists $c > 0$ satisfying the property that, for all $f \in \mathcal{S}$ (or, equivalently, for all $f \in \mathbb{D}^{p,2}$),

$$\left| \int_{\mathbb{R}} f^{(p)}(x) g(x) d\gamma(x) \right| \leq c \sqrt{\int_{\mathbb{R}} f^2(x) d\gamma(x)}. \tag{1.2.1}$$

Fix $g \in \mathrm{Dom}\,\delta^p$. Since condition (1.2.1) holds, the linear operator $f \mapsto \int_{\mathbb{R}} f^{(p)}(x) g(x) d\gamma(x)$ is continuous from $\mathcal{S}$, equipped with the $L^2(\gamma)$-norm, into $\mathbb{R}$. Thus, we can extend this operator to a linear operator from $L^2(\gamma)$ into $\mathbb{R}$. By the Riesz representation theorem, there exists a unique element in $L^2(\gamma)$, denoted by $\delta^p g$, such that $\int_{\mathbb{R}} f^{(p)}(x) g(x) d\gamma(x) = \int_{\mathbb{R}} f(x) \delta^p g(x) d\gamma(x)$ for all $f \in \mathcal{S}$.

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Definition 1.2.2 Fix an integer $p \geq 1$. The pth **divergence operator** δ^p is defined as follows. If $g \in \operatorname{Dom}\delta^p$, then $\delta^p g$ is the unique element of $L^2(\gamma)$ characterized by the following duality formula: for all $f \in \mathcal{S}$ (or, equivalently, for all $f \in \mathbb{D}^{p,2}$),

$$\int_{\mathbb{R}} f^{(p)}(x)g(x)d\gamma(x) = \int_{\mathbb{R}} f(x)\delta^p g(x)d\gamma(x). \tag{1.2.2}$$

When $p = 1$, we shall often write δ instead of δ^1.

Remark 1.2.3 Taking f to be equal to a constant in (1.2.2), we deduce that, for every $p \geq 1$ and every $g \in \operatorname{Dom}\delta^p$,

$$\int_{\mathbb{R}} \delta^p g(x)d\gamma(x) = 0. \tag{1.2.3}$$

Notice that the operator δ^p is closed (being the adjoint of D^p). Also,

$$\delta^p g = \delta(\delta^{p-1}g) = \delta^{p-1}(\delta g) \tag{1.2.4}$$

for every $g \in \operatorname{Dom}\delta^p$. In particular, the first equality in (1.2.4) implies that, if $g \in \operatorname{Dom}\delta^p$, then $\delta^{p-1}g \in \operatorname{Dom}\delta$, whereas from the second equality we infer that, if $g \in \operatorname{Dom}\delta^p$, then $\delta g \in \operatorname{Dom}\delta^{p-1}$.

Exercise 1.2.4 Prove the two equalities in (1.2.4).

For every $f, g \in \mathcal{S}$, we can write, by virtue of Lemma 1.1.1,

$$\int_{\mathbb{R}} f'(x)g(x)d\gamma(x) = \int_{\mathbb{R}} xf(x)g(x)d\gamma(x) - \int_{\mathbb{R}} f(x)g'(x)d\gamma(x). \tag{1.2.5}$$

Relation (1.2.5) implies that $\mathcal{S} \subset \operatorname{Dom}\delta$ and, for $g \in \mathcal{S}$, that $\delta g(x) = xg(x) - g'(x)$. By approximation, we deduce that $\mathbb{D}^{1,2} \subset \operatorname{Dom}\delta$, and also that the previous formula for δg continues to hold when $g \in \mathbb{D}^{1,2}$, that is, $\delta g = G - Dg$ for every $g \in \mathbb{D}^{1,2}$, where $G(x) = xg(x)$. More generally, we can prove that $\mathbb{D}^{p,2} \subset \operatorname{Dom}\delta^p$ for any $p \geq 1$.

1.3 Ornstein–Uhlenbeck operators

Definition 1.3.1 The **Ornstein–Uhlenbeck semigroup**, written $(P_t)_{t\geq 0}$, is defined as follows. For $f \in \mathcal{S}$ and $t \geq 0$,

$$P_t f(x) = \int_{\mathbb{R}} f(e^{-t}x + \sqrt{1-e^{-2t}}y)d\gamma(y), \quad x \in \mathbb{R}. \tag{1.3.1}$$

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The semigroup characterization is proved in Proposition 1.3.3. An explicit connection with Ornstein–Uhlenbeck stochastic processes is provided in Exercise 1.7.4.

Plainly, $P_0 f(x) = f(x)$. By using the fact that f is an element of $\mathcal{S}$ and by dominated convergence, it is immediate that $P_\infty f(x) := \lim_{t\to\infty} P_t f(x) = \int_{\mathbb{R}} f(y)d\gamma(y)$. On the other hand, by applying the Jensen inequality to the right-hand side of (1.3.1), we infer that, for every $q \in [1, \infty)$,

$$\begin{aligned}\int_{\mathbb{R}} |P_t f(x)|^q d\gamma(x) &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(e^{-t}x + \sqrt{1-e^{-2t}}y)d\gamma(y) \right|^q d\gamma(x) \\ &\le \int_{\mathbb{R}^2} |f(e^{-t}x + \sqrt{1-e^{-2t}}y)|^q d\gamma(x)d\gamma(y) \\ &= \int_{\mathbb{R}} |f(x)|^q d\gamma(x). \end{aligned} \tag{1.3.2}$$

The last equality in (1.3.2) follows from the well-known (and easily checked using the characteristic function) fact that, if N, N' are two independent standard Gaussian random variables, then $e^{-t}N + \sqrt{1-e^{-2t}}N'$ is also standard Gaussian.

The relations displayed in (1.3.2), together with Proposition 1.1.5, show that the expression on the right-hand side of (1.3.1) is indeed well defined for $f \in L^q(\gamma)$, $q \ge 1$. Moreover, a contraction property holds.

Proposition 1.3.2 *For every $t \ge 0$ and every $q \in [1, \infty)$, P_t extends to a linear contraction operator on $L^q(\gamma)$.*

As anticipated, the fundamental property of the class $(P_t)_{t\ge0}$ is that it is a semigroup of operators.

Proposition 1.3.3 *For any $s, t \ge 0$, we have $P_t P_s = P_{t+s}$ on $L^1(\gamma)$.*

Proof For all $f \in L^1(\gamma)$, we can write

$$\begin{aligned} P_t P_s f(x) &= \int_{\mathbb{R}^2} f(e^{-s-t}x + e^{-s}\sqrt{1-e^{-2t}}y + \sqrt{1-e^{-2s}}z)d\gamma(y)d\gamma(z) \\ &= \int_{\mathbb{R}} f(e^{-s-t}x + \sqrt{1-e^{-2s-2t}}y)d\gamma(y) = P_{t+s}f(x), \end{aligned}$$

where the second inequality follows from the easily verified fact that, if N, N' are two independent standard Gaussian random variables, then $e^{-s}\sqrt{1-e^{-2t}}N + \sqrt{1-e^{-2s}}N'$ and $\sqrt{1-e^{-2(s+t)}}N$ have the same law. □

The following result shows that P_t and D can be interchanged on $\mathbb{D}^{1,2}$.

Proposition 1.3.4 *Let $f \in \mathbb{D}^{1,2}$ and $t \ge 0$. Then $P_t f \in \mathbb{D}^{1,2}$ and $DP_t f = e^{-t}P_t Df$.*

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