1

Polarity

1.1 Polar hypersurfaces

1.1.1 The polar pairing

We will take $\mathbb{C}$ as the base field, although many constructions in this book work over an arbitrary algebraically closed field.

We will usually denote by E a vector space of dimension $n+1$. Its dual vector space will be denoted by $E^\vee$.

Let $S(E)$ be the *symmetric algebra* of E, the quotient of the tensor algebra $T(E) = \oplus_{d\geq 0} E^{\otimes d}$ by the two-sided ideal generated by tensors of the form $v \otimes w - w \otimes v$, $v, w \in E$. The symmetric algebra is a graded commutative algebra, its graded components $S^d(E)$ are the images of $E^{\otimes d}$ in the quotient. The vector space $S^d(E)$ is called the dth *symmetric power* of E. Its dimension is equal to $\binom{d+n}{n}$. The image of a tensor $v_1 \otimes \cdots \otimes v_d$ in $S^d(E)$ is denoted by $v_1 \cdots v_d$.

The permutation group $\mathfrak{S}_d$ has a natural linear representation in $E^{\otimes d}$ via permuting the factors. The symmetrization operator $\oplus_{\sigma\in\mathfrak{S}_d}\sigma$ is a projection operator onto the subspace of symmetric tensors $S_d(E) = (E^{\otimes d})^{\mathfrak{S}_d}$ multiplied by $d!$. It factors through $S^d(E)$ and defines a natural isomorphism

$$S^d(E) \to S_d(E).$$

Replacing E by its dual space $E^\vee$, we obtain a natural isomorphism

$$\mathfrak{p}_d : S^d(E^\vee) \to S_d(E^\vee). \tag{1.1}$$

Under the identification of $(E^\vee)^{\otimes d}$ with the space $(E^{\otimes d})^\vee$, we will be able to identify $S_d(E^\vee)$ with the space $\mathrm{Hom}(E^d, \mathbb{C})^{\mathfrak{S}_d}$ of symmetric d-multilinear functions $E^d \to \mathbb{C}$. The isomorphism $\mathfrak{p}_d$ is classically known as the *total polarization map*.

Next we use the fact that the quotient map $E^{\otimes d} \to S^d(E)$ is a universal symmetric d-multilinear map, i.e. any linear map $E^{\otimes d} \to F$ with values in some vector space F factors through a linear map $S^d(E) \to F$. If $F = \mathbb{C}$, this gives a natural isomorphism

$$S_d(E^\vee) \to S^d(E)^\vee.$$

Composing it with $\mathfrak{p}_d$, we get a natural isomorphism

$$S^d(E^\vee) \to S^d(E)^\vee. \tag{1.2}$$

It can be viewed as a perfect bilinear pairing, the *polar pairing*

$$\langle , \rangle : S^d(E^\vee) \otimes S^d(E) \to \mathbb{C}. \tag{1.3}$$

This pairing extends the natural pairing between E and $E^\vee$ to the symmetric powers. Explicitly,

$$\langle l_1 \cdots l_d, w_1 \cdots w_d \rangle = \sum_{\sigma \in \mathfrak{S}_d} l_{\sigma^{-1}(1)}(w_1) \cdots l_{\sigma^{-1}(d)}(w_d).$$

One can extend the total polarization isomorphism to a *partial polarization map*

$$\langle , \rangle : S^d(E^\vee) \otimes S^k(E) \to S^{d-k}(E^\vee), \quad k \le d, \tag{1.4}$$

$$\langle l_1 \cdots l_d, w_1 \cdots w_k \rangle = \sum_{1 \le i_1 \le \cdots \le i_k \le n} \langle l_{i_1} \cdots l_{i_k}, w_1 \cdots w_k \rangle \prod_{j \ne i_1, \ldots, i_k} l_j.$$

In coordinates, if we choose a basis $(\xi_0, \ldots, \xi_n)$ in E and its dual basis $t_0, \ldots, t_n$ in $E^\vee$, then we can identify $S(E^\vee)$ with the polynomial algebra $\mathbb{C}[t_0, \ldots, t_n]$ and $S^d(E^\vee)$ with the space $\mathbb{C}[t_0, \ldots, t_n]_d$ of homogeneous polynomials of degree d. Similarly, we identify $S^d(E)$ with $\mathbb{C}[\xi_0, \ldots, \xi_n]$. The polarization isomorphism extends by linearity of the pairing on monomials

$$\langle t_0^{i_0} \cdots t_n^{i_n}, \xi_0^{j_0} \cdots \xi_n^{j_n} \rangle = \begin{cases} i_0! \cdots i_n! & \text{if } (i_0, \ldots, i_n) = (j_0, \ldots, j_n), \\ 0 & \text{otherwise.} \end{cases}$$

One can give an explicit formula for pairing (1.4) in terms of differential operators. Since $\langle t_i, \xi_j \rangle = \delta_{ij}$, it is convenient to view a basis vector ξ_j as the partial derivative operator $\partial_j = \frac{\partial}{\partial t_j}$. Hence any element $\psi \in S^k(E) = \mathbb{C}[\xi_0, \ldots, \xi_n]_k$ can be viewed as a differential operator

$$D_\psi = \psi(\partial_0, \ldots, \partial_n).$$

The pairing (1.4) becomes

$$\langle \psi(\xi_0, \ldots, \xi_n), f(t_0, \ldots, t_n) \rangle = D_\psi(f).$$

For any monomial $\boldsymbol{\partial}^{\mathbf{i}} = \partial_0^{i_0} \cdots \partial_n^{i_n}$ and any monomial $\mathbf{t}^{\mathbf{j}} = t_0^{j_0} \cdots t_n^{j_n}$, we have

$$\boldsymbol{\partial}^{\mathbf{i}}(\mathbf{t}^{\mathbf{j}}) = \begin{cases} \frac{\mathbf{j}!}{(\mathbf{j}-\mathbf{i})!} \mathbf{t}^{\mathbf{j}-\mathbf{i}} & \text{if } \mathbf{j} - \mathbf{i} \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

Here and later we use the vector notation:

$$\mathbf{i}! = i_0! \cdots i_n!, \quad \binom{k}{\mathbf{i}} = \frac{k!}{\mathbf{i}!}, \quad |\mathbf{i}| = i_0 + \cdots + i_n.$$

The total polarization $\tilde{f}$ of a polynomial f is given explicitly by the following formula:

$$\tilde{f}(v_1, \dots, v_d) = D_{v_1 \cdots v_d}(f) = (D_{v_1} \circ \cdots \circ D_{v_d})(f).$$

Taking $v_1 = \cdots = v_d = v$, we get

$$\tilde{f}(v, \dots, v) = d! f(v) = D_{v^d}(f) = \sum_{|\mathbf{i}|=d} \binom{d}{\mathbf{i}} \mathbf{a}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}} f. \tag{1.6}$$

Remark 1.1.1 The polarization isomorphism was known in the classical literature as the *symbolic method*. Suppose $f = l^d$ is a dth power of a linear form. Then $D_v(f) = dl(v)^{d-1}$ and

$$D_{v_1} \circ \cdots \circ D_{v_k}(f) = d(d-1)\cdots(d-k+1) l(v_1)\cdots l(v_k) l^{d-k}.$$

In classical notation, a linear form $\sum a_i x_i$ on $\mathbb{C}^{n+1}$ is denoted by a_x and the dot-product of two vectors a, b is denoted by (ab). Symbolically, one denotes any homogeneous form by a_x^d and the right-hand side of the previous formula reads as $d(d-1)\cdots(d-k+1)(ab)^k a_x^{d-k}$.

Let us take $E = S^m(U^\vee)$ for some vector space U and consider the linear space $S^d(S^m(U^\vee)^\vee)$. Using the polarization isomorphism, we can identify $S^m(U^\vee)^\vee$ with $S^m(U)$. Let $(\xi_0, \dots, \xi_r)$ be a basis in U and $(t_0, \dots, t_{r+1})$ be the dual basis in $U^\vee$. Then we can take for a basis of $S^m(U)$ the monomials $\boldsymbol{\xi}^{\mathbf{j}}$. The dual basis in $S^m(U^\vee)$ is formed by the monomials $\frac{1}{\mathbf{i}!}\mathbf{x}^{\mathbf{i}}$. Thus, for any $f \in S^m(U^\vee)$, we can write

$$m! f = \sum_{|\mathbf{i}|=m} \binom{m}{\mathbf{i}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}. \tag{1.7}$$

In symbolic form, $m! f = (a_x)^m$. Consider the matrix

$$\Xi = \begin{pmatrix} \xi_0^{(1)} & \dots & \xi_0^{(d)} \\ \vdots & \vdots & \vdots \\ \xi_r^{(1)} & \dots & \xi_r^{(d)} \end{pmatrix},$$

where $(\xi_0^{(k)}, \dots, \xi_r^{(k)})$ is a copy of a basis in U. Then the space $S^d(S^m(U))$ is equal to the subspace of the polynomial algebra $\mathbb{C}[(\xi_j^{(i)})]$ in $d(r+1)$ variables $\xi_j^{(i)}$ of polynomials that are homogeneous of degree m in each column of the matrix and symmetric with respect to permutations of the columns. Let $J \subset \{1, \dots, d\}$ with $\#J = r+1$ and (J) be the corresponding maximal minor of the matrix Ξ. Assume $r+1$ divides dm. Consider a product of $k = \frac{dm}{r+1}$ such minors in which each column participates exactly m times. Then a sum of such products that is invariant with respect to permutations of columns represents an element from $S^d(S^m(U))$ that has an additional property that it is invariant with respect to the group $\mathrm{SL}(U) \cong \mathrm{SL}(r+1, \mathbb{C})$ which acts on U by the left multiplication with a vector $(\xi_0, \dots, \xi_r)$. The *First Fundamental Theorem* of invariant theory states that any element in $S^d(S^m(U))^{\mathrm{SL}(U)}$ is obtained in this way (see [180]). We can interpret elements of $S^d(S^m(U^\vee)^\vee)$ as polynomials in coefficients of $a_{\mathbf{i}}$

of a homogeneous form of degree d in $r+1$ variables written in the form (1.7). We write symbolically an invariant in the form $(J_1)\cdots(J_k)$, meaning that it is obtained as a sum of such products with some coefficients. If the number d is small, we can use letters, say $a, b, c, \ldots$, instead of numbers $1, \ldots, d$. For example, $(12)^2(13)^2(23)^2 = (ab)^2(bc)^2(ac)^2$ represents an element in $S^3(S^4(\mathbb{C}^2))$.

In a similar way, one considers the matrix

$$\begin{pmatrix} \xi_0^{(1)} & \ldots & \xi_0^{(d)} & t_0^{(1)} & \ldots & t_0^{(s)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_r^{(1)} & \ldots & \xi_r^{(d)} & t_r^{(1)} & \ldots & t_r^{(s)} \end{pmatrix}.$$

The product of k maximal minors such that each of the first d columns occurs exactly k times and each of the last s columns occurs exactly p times represents a *covariant* of degree p and order k. For example, $(ab)^2 a_x b_x$ represents the *Hessian determinant*

$$\mathrm{He}(f) = \det \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix}$$

of a binary cubic form f.

The *projective space* of lines in E will be denoted by $|E|$. The space $|E^\vee|$ will be denoted by $\mathbb{P}(E)$ (following Grothendieck's notation). We call $\mathbb{P}(E)$ the *dual projective space* of $|E|$. We will often denote it by $|E|^\vee$.

A basis $\xi_0, \ldots, \xi_n$ in E defines an isomorphism $E \cong \mathbb{C}^{n+1}$ and identifies $|E|$ with the projective space $\mathbb{P}^n := |\mathbb{C}^{n+1}|$. For any nonzero vector $v \in E$ we denote by $[v]$ the corresponding point in $|E|$. If $E = \mathbb{C}^{n+1}$ and $v = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ we set $[v] = [a_0, \ldots, a_n]$. We call $[a_0, \ldots, a_n]$ the *projective coordinates* of a point $[a] \in \mathbb{P}^n$. Other common notation for the projective coordinates of $[a]$ is $(a_0 : a_1 : \ldots : a_n)$, or simply $(a_0, \ldots, a_n)$, if no confusion arises.

The projective space comes with the tautological invertible sheaf $\mathcal{O}_{|E|}(1)$ whose space of global sections is identified with the dual space $E^\vee$. Its dth tensor power is denoted by $\mathcal{O}_{|E|}(d)$. Its space of global sections is identified with the symmetric dth power $S^d(E^\vee)$.

For any $f \in S^d(E^\vee), d > 0$, we denote by $V(f)$ the corresponding effective divisor from $|\mathcal{O}_{|E|}(d)|$, considered as a closed subscheme of $|E|$, not necessary reduced. We call $V(f)$ a *hypersurface* of degree d in $|E|$ defined by equation $f = 0$.[1] A hypersurface of degree 1 is a *hyperplane*. By definition, $V(0) = |E|$ and $V(1) = \emptyset$. The projective space $|S^d(E^\vee)|$ can be viewed as the projective space of hypersurfaces in $|E|$. It is equal to the complete linear system $|\mathcal{O}_{|E|}(d)|$. Using isomorphism (1.2), we may identify the projective space $|S^d(E)|$ of hypersurfaces of degree d in $|E^\vee|$ with the dual of the projective space $|S^d E^\vee|$. A hypersurface of degree d in $|E^\vee|$ is classically known as an *envelope* of *class* d.

[1] This notation should not be confused with the notation of the closed subset in Zariski topology defined by the ideal (f). It is equal to $V(f)_{\mathrm{red}}$.

The natural isomorphisms

$$(E^\vee)^{\otimes d} \cong H^0(|E|^d, \mathcal{O}_{|E|}(1)^{\boxtimes d}),\ S_d(E^\vee) \cong H^0(|E|^d, \mathcal{O}_{|E|}(1)^{\boxtimes d})^{\mathfrak{S}_d}$$

allow one to give the following geometric interpretation of the polarization isomorphism. Consider the diagonal embedding $\delta_d : |E| \hookrightarrow |E|^d$. Then the total polarization map is the inverse of the isomorphism

$$\delta_d^* : H^0(|E|^d, \mathcal{O}_{|E|}(1)^{\boxtimes d})^{\mathfrak{S}_d} \to H^0(|E|, \mathcal{O}_{|E|}(d)).$$

We view $a_0\partial_0 + \cdots + a_n\partial_n \neq 0$ as a point $a \in |E|$ with projective coordinates $[a_0, \ldots, a_n]$.

Definition 1.1.2 *Let $X = V(f)$ be a hypersurface of degree d in $|E|$ and $x = [v]$ be a point in $|E|$. The hypersurface*

$$P_{a^k}(X) := V(D_{v^k}(f))$$

of degree $d-k$ is called the kth polar hypersurface *of the point a with respect to the hypersurface $V(f)$ (or of the hypersurface with respect to the point).*

Example 1.1.3 Let $d = 2$, i.e.

$$f = \sum_{i=0}^{n} \alpha_{ii} t_i^2 + 2 \sum_{0 \le i < j \le n} \alpha_{ij} t_i t_j$$

is a quadratic form on $\mathbb{C}^{n+1}$. For any $x = [a_0, \ldots, a_n] \in \mathbb{P}^n$, $P_x(V(f)) = V(g)$, where

$$g = \sum_{i=0}^{n} a_i \frac{\partial f}{\partial t_i} = 2 \sum_{0 \le i < j \le n} a_i \alpha_{ij} t_j, \quad \alpha_{ji} = \alpha_{ij}.$$

The linear map $v \mapsto D_v(f)$ is a map from $\mathbb{C}^{n+1}$ to $(\mathbb{C}^{n+1})^\vee$ which can be identified with the *polar bilinear form* associated to f with matrix $2(\alpha_{ij})$.

Let us give another definition of the polar hypersurfaces $P_{x^k}(X)$. Choose two different points $a = [a_0, \ldots, a_n]$ and $b = [b_0, \ldots, b_n]$ in $\mathbb{P}^n$ and consider the line $\ell = \overline{ab}$ spanned by the two points as the image of the map

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^n, \quad [u_0, u_1] \mapsto u_0 a + u_1 b := [a_0 u_0 + b_0 u_1, \ldots, a_n u_0 + b_n u_1]$$

(a parametric equation of ℓ). The intersection $\ell \cap X$ is isomorphic to the positive divisor on $\mathbb{P}^1$ defined by the degree d homogeneous form

$$\varphi^*(f) = f(u_0 a + u_1 b) = f(a_0 u_0 + b_0 u_1, \ldots, a_n u_0 + b_n u_1).$$

Using the Taylor formula at $(0,0)$, we can write

$$\varphi^*(f) = \sum_{k+m=d} \frac{1}{k!m!} u_0^k u_1^m A_{km}(a, b), \tag{1.8}$$

where

$$A_{km}(a,b) = \frac{\partial^d \varphi^*(f)}{\partial u_0^k \partial u_1^m}(0,0).$$

Using the Chain Rule, we get

$$A_{km}(a,b) = \sum_{|\mathbf{i}|=k, |\mathbf{j}|=m} \binom{k}{\mathbf{i}}\binom{m}{\mathbf{j}} \mathbf{a}^{\mathbf{i}}\mathbf{b}^{\mathbf{j}}\boldsymbol{\partial}^{\mathbf{i}+\mathbf{j}} f = D_{a^k b^m}(f). \tag{1.9}$$

Observe the symmetry

$$A_{km}(a,b) = A_{mk}(b,a). \tag{1.10}$$

When we fix a and let b vary in $\mathbb{P}^n$ we obtain a hypersurface $V(A(a,x))$ of degree $d-k$ which is the kth polar hypersurface of $X = V(f)$ with respect to the point a. When we fix b and vary a in $\mathbb{P}^n$, we obtain the mth polar hypersurface $V(A(x,b))$ of X with respect to the point b.

Note that

$$D_{a^k b^m}(f) = D_{a^k}(D_{b^m}(f)) = D_{b^m}(a) = D_{b^m}(D_{a^k}(f)) = D_{a^k}(f)(b). \tag{1.11}$$

This gives the symmetry property of polars

$$b \in P_{a^k}(X) \Leftrightarrow a \in P_{b^{d-k}}(X). \tag{1.12}$$

Since we are in characteristic 0, if $m \le d$, $D_{a^m}(f)$ cannot be zero for all a. To see this we use the *Euler formula*:

$$df = \sum_{i=0}^{n} t_i \frac{\partial f}{\partial t_i}.$$

Applying this formula to the partial derivatives, we obtain

$$d(d-1)\dots(d-k+1)f = \sum_{|\mathbf{i}|=k} \binom{k}{\mathbf{i}} \mathbf{t}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}} f \tag{1.13}$$

(also called the Euler formula). It follows from this formula that, for all $k \le d$,

$$a \in P_{a^k}(X) \Leftrightarrow a \in X. \tag{1.14}$$

This is known as the *reciprocity theorem*.

Example 1.1.4 Let M_d be the vector space of complex square matrices of size d with coordinates t_{ij}. We view the determinant function $\det : M_d \to \mathbb{C}$ as an element of $S^d(M_d^\vee)$, i.e. a polynomial of degree d in the variables t_{ij}. Let $C_{ij} = \frac{\partial \det}{\partial t_{ij}}$. For any point $A = (a_{ij})$ in M_d the value of C_{ij} at A is equal to the ijth cofactor of A. Applying (1.6), for any $B = (b_{ij}) \in M_d$, we obtain

$$D_{A^{d-1}B}(\det) = D_A^{d-1}(D_B(\det)) = D_A^{d-1}\left(\sum b_{ij}C_{ij}\right) = (d-1)! \sum b_{ij}C_{ij}(A).$$

Thus $D_A^{d-1}(\det)$ is a linear function $\sum t_{ij}C_{ij}$ on M_d. The linear map

$$S^{d-1}(M_n) \to M_d^\vee, \quad A \mapsto \frac{1}{(d-1)!} D_A^{d-1}(\det),$$

can be identified with the function $A \mapsto \mathrm{adj}(A)$, where $\mathrm{adj}(A)$ is the cofactor matrix (classically called the *adjugate matrix* of A, but not the adjoint matrix as it is often called in modern text-books).

1.1.2 First polars

Let us consider some special cases. Let $X = V(f)$ be a hypersurface of degree d. Obviously, any 0th polar of X is equal to X and, by (1.12), the dth polar $P_{a^d}(X)$ is empty if $a \notin X$ and equals $\mathbb{P}^n$ if $a \in X$. Now take $k = 1, d-1$. Using (1.6), we obtain

$$D_a(f) = \sum_{i=0}^{n} a_i \frac{\partial f}{\partial t_i},$$

$$\frac{1}{(d-1)!} D_{a^{d-1}}(f) = \sum_{i=0}^{n} \frac{\partial f}{\partial t_i}(a) t_i.$$

Together with (1.12) this implies the following.

Theorem 1.1.5 *For any smooth point $x \in X$, we have*

$$P_{x^{d-1}}(X) = \mathbb{T}_x(X).$$

If x is a singular point of X, $P_{x^{d-1}}(X) = \mathbb{P}^n$. Moreover, for any $a \in \mathbb{P}^n$,

$$X \cap P_a(X) = \{x \in X : a \in \mathbb{T}_x(X)\}.$$

Here and later on we denote by $\mathbb{T}_x(X)$ the *embedded tangent space* of a projective subvariety $X \subset \mathbb{P}^n$ at its point x. It is a linear subspace of $\mathbb{P}^n$ equal to the projective closure of the affine Zariski tangent space $T_x(X)$ of X at x (see [275], p. 181).

In classical terminology, the intersection $X \cap P_a(X)$ is called the *apparent boundary* of X from the point a. If one projects X to $\mathbb{P}^{n-1}$ from the point a, then the apparent boundary is the ramification divisor of the projection map.

The following picture (Figure 1.1) makes an attempt to show what happens in the case when X is a conic.

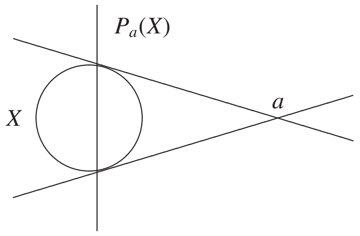

Figure 1.1 Polar line of a conic.

The set of first polars $P_a(X)$ defines a linear system contained in the complete linear system $\left|\mathcal{O}_{\mathbb{P}^n}(d-1)\right|$. The dimension of this linear system is $\le n$. We will be freely using the language of linear systems and divisors on algebraic varieties (see [279]).

Proposition 1.1.6 *The dimension of the linear system of first polars is $\le r$ if and only if, after a linear change of variables, the polynomial f becomes a polynomial in $r+1$ variables.*

Proof Let $X = V(f)$. It is obvious that the dimension of the linear system of first polars is $\le r$ if and only if the linear map $E \to S^{d-1}(E^\vee), v \mapsto D_v(f)$ has kernel of dimension $\ge n-r$. Choosing an appropriate basis, we may assume that the kernel is generated by vectors $(1, 0, \ldots, 0)$, etc. Now, it is obvious that f does not depend on the variables $t_0, \ldots, t_{n-r-1}$. □

It follows from Theorem 1.1.5 that the first polar $P_a(X)$ of a point a with respect to a hypersurface X passes through all singular points of X. One can say more.

Proposition 1.1.7 *Let a be a singular point of X of multiplicity m. For each $r \le \deg X - m$, $P_{a^r}(X)$ has a singular point at a of multiplicity m and the tangent cone of $P_{a^r}(X)$ at a coincides with the tangent cone $\mathrm{TC}_a(X)$ of X at a. For any point $b \ne a$, the rth polar $P_{b^r}(X)$ has multiplicity $\ge m-r$ at a and its tangent cone at a is equal to the rth polar of $\mathrm{TC}_a(X)$ with respect to b.*

Proof Let us prove the first assertion. Without loss of generality, we may assume that $a = [1, 0, \ldots, 0]$. Then $X = V(f)$, where

$$f = t_0^{d-m} f_m(t_1, \ldots, t_n) + t_0^{d-m-1} f_{m+1}(t_1, \ldots, t_n) + \cdots + f_d(t_1, \ldots, t_n). \tag{1.15}$$

The equation $f_m(t_1, \ldots, t_n) = 0$ defines the tangent cone of X at b. The equation of $P_{a^r}(X)$ is

$$\frac{\partial^r f}{\partial t_0^r} = r! \sum_{i=0}^{d-m-r} \binom{d-m-i}{r} t_0^{d-m-r-i} f_{m+i}(t_1, \ldots, t_n) = 0.$$

It is clear that $[1, 0, \ldots, 0]$ is a singular point of $P_{a^r}(X)$ of multiplicity m with the tangent cone $V(f_m(t_1, \ldots, t_n))$.

Now we prove the second assertion. Without loss of generality, we may assume that $a = [1, 0, \ldots, 0]$ and $b = [0, 1, 0, \ldots, 0]$. Then the equation of $P_{b^r}(X)$ is

$$\frac{\partial^r f}{\partial t_1^r} = t_0^{d-m} \frac{\partial^r f_m}{\partial t_1^r} + \cdots + \frac{\partial^r f_d}{\partial t_1^r} = 0.$$

The point a is a singular point of multiplicity $\ge m-r$. The tangent cone of $P_{b^r}(X)$ at the point a is equal to $V(\frac{\partial^r f_m}{\partial t_1^r})$ and this coincides with the rth polar of $\mathrm{TC}_a(X) = V(f_m)$ with respect to b. □

We leave it to the reader to see what happens if $r > d - m$.

Keeping the notation from the previous proposition, consider a line ℓ through the point a such that it intersects X at some point $x \neq a$ with multiplicity larger than one. The closure $\mathrm{EC}_a(X)$ of the union of such lines is called the *enveloping cone* of X at the point a. If X is not a cone with vertex at a, the branch divisor of the projection $p : X \setminus \{a\} \to \mathbb{P}^{n-1}$ from a is equal to the projection of the enveloping cone. Let us find the equation of the enveloping cone.

As above, we assume that $a = [1, 0, \ldots, 0]$. Let H be the hyperplane $t_0 = 0$. Write ℓ in a parametric form $ua + vx$ for some $x \in H$. Plugging in equation (1.15), we get

$$P(t) = t^{d-m} f_m(x_1, \ldots, x_n) + t^{d-m-1} f_{m+1}(x_1, \ldots, x_m) + \cdots + f_d(x_1, \ldots, x_n) = 0,$$

where $t = u/v$.

We assume that $X \neq \mathrm{TC}_a(X)$, i.e. X is not a cone with a vertex at a (otherwise, by definition, $\mathrm{EC}_a(X) = \mathrm{TC}_a(X)$). The image of the tangent cone under the projection $p : X \setminus \{a\} \to H \cong \mathbb{P}^{n-1}$ is a proper closed subset of H. If $f_m(x_1, \ldots, x_n) \neq 0$, then a multiple root of $P(t)$ defines a line in the enveloping cone. Let $\mathsf{D}_k(A_0, \ldots, A_k)$ be the discriminant of a general polynomial $P = A_0 T^k + \cdots + A_k$ of degree k. Recall that

$$A_0 \mathsf{D}_k(A_0, \ldots, A_k) = (-1)^{k(k-1)/2} \mathrm{Res}(P, P')(A_0, \ldots, A_k),$$

where $\mathrm{Res}(P, P')$ is the resultant of P and its derivative P'. It follows from the known determinant expression of the resultant that

$$\mathsf{D}_k(0, A_1, \ldots, A_k) = (-1)^{\frac{k^2-k+2}{2}} A_0^2 \mathsf{D}_{k-1}(A_1, \ldots, A_k).$$

The equation $P(t) = 0$ has a multiple zero with $t \neq 0$ if and only if

$$\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x)) = 0.$$

So, we see that

$$\mathrm{EC}_a(X) \subset V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x))), \tag{1.16}$$
$$\mathrm{EC}_a(X) \cap \mathrm{TC}_a(X) \subset V(\mathsf{D}_{d-m-1}(f_{m+1}(x), \ldots, f_d(x))).$$

It follows from the computation of $\frac{\partial^r f}{\partial t_0^r}$ in the proof of the previous Proposition that the hypersurface $V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x)))$ is equal to the projection of $P_a(X) \cap X$ to H.

Suppose $V(\mathsf{D}_{d-m-1}(f_{m+1}(x), \ldots, f_d(x)))$ and $\mathrm{TC}_a(X)$ do not share an irreducible component. Then

$$V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x))) \setminus \mathrm{TC}_a(X) \cap V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x)))$$

$$= V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x))) \setminus V(\mathsf{D}_{d-m-1}(f_{m+1}(x), \ldots, f_d(x))) \subset \mathrm{EC}_a(X),$$

gives the opposite inclusion of (1.16), and we get

$$\mathrm{EC}_a(X) = V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x))). \tag{1.17}$$

Note that the discriminant $\mathsf{D}_{d-m}(A_0, \ldots, A_k)$ is an invariant of the group SL(2) in its natural representation on degree k binary forms. Taking the diagonal subtorus, we immediately infer that any monomial $A_0^{i_0} \cdots A_k^{i_k}$ entering in the discriminant polynomial satisfies

$$k \sum_{s=0}^{k} i_s = 2 \sum_{s=0}^{k} s i_s.$$

It is also known that the discriminant is a homogeneous polynomial of degree $2k - 2$. Thus, we get

$$k(k-1) = \sum_{s=0}^{k} s i_s.$$

In our case $k = d - m$, we obtain that

$$\deg V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x))) = \sum_{s=0}^{d-m} (m+s) i_s$$

$$= m(2d - 2m - 2) + (d-m)(d-m-1) = (d+m)(d-m-1).$$

This is the expected degree of the enveloping cone.

Example 1.1.8 Assume $m = d - 2$, then

$$\mathsf{D}_2(f_{d-2}(x), f_{d-1}(x), f_d(x)) = f_{d-1}(x)^2 - 4 f_{d-2}(x) f_d(x),$$
$$\mathsf{D}_2(0, f_{d-1}(x), f_d(x)) = f_{d-2}(x) = 0.$$

Suppose $f_{d-2}(x)$ and f_{d-1} are coprime. Then our assumption is satisfied, and we obtain

$$\mathrm{EC}_a(X) = V(f_{d-1}(x)^2 - 4 f_{d-2}(x) f_d(x)).$$

Observe that the hypersurfaces $V(f_{d-2}(x))$ and $V(f_d(x))$ are everywhere tangent to the enveloping cone. In particular, the quadric tangent cone $\mathrm{TC}_a(X)$ is everywhere tangent to the enveloping cone along the intersection of $V(f_{d-2}(x))$ with $V(f_{d-1}(x))$.

For any nonsingular quadric Q, the map $x \mapsto P_x(Q)$ defines a projective isomorphism from the projective space to the dual projective space. This is a special case of a correlation.

According to classical terminology, a projective automorphism of $\mathbb{P}^n$ is called a *collineation*. An isomorphism from $|E|$ to its dual space $\mathbb{P}(E)$ is called a *correlation*. A correlation $\mathfrak{c} : |E| \to \mathbb{P}(E)$ is given by an invertible linear map $\phi : E \to E^\vee$ defined uniquely up to proportionality. A correlation transforms points in $|E|$ to hyperplanes in $|E|$. A point $x \in |E|$ is called *conjugate* to a point $y \in |E|$ with respect to the correlation $\mathfrak{c}$ if $y \in \mathfrak{c}(x)$. The transpose of the inverse map ${}^t\phi^{-1} : E^\vee \to E$ transforms hyperplanes in $|E|$ to points in $|E|$. It can be considered as a correlation between the dual spaces $\mathbb{P}(E)$ and $|E|$. It is denoted by $\mathfrak{c}^\vee$