1

Jordan and Lie theory

1.1 Jordan algebras

We begin by discussing the basic structures and some examples of Jordan algebras which are relevant in later chapters. One important feature is that multiplication in these algebras need not be associative.

By an *algebra* we mean a vector space $\mathcal{A}$ over a field, equipped with a bilinear product $(a, b) \in \mathcal{A}^2 \mapsto ab \in \mathcal{A}$. We do not assume associativity of the product. If the product is associative, we call $\mathcal{A}$ *associative*.

Homomorphisms and isomorphisms between two algebras are defined as in the case of associative algebras. An *antiautomorphism* of an algebra $\mathcal{A}$ is a linear bijection $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ such that $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in \mathcal{A}$.

We call an algebra $\mathcal{A}$ *unital* if it contains an identity, which will always be denoted by $\mathbf{1}$, unless stated otherwise. As usual, one can adjoint an identity $\mathbf{1}$ to a nonunital algebra $\mathcal{A}$ to form a unital algebra $\mathcal{A}_1$, called the *unit extension* of $\mathcal{A}$.

A *Jordan algebra* is a commutative algebra over a field $\mathbb{F}$, and satisfies the Jordan identity

$$(ab)a^2 = a(ba^2) \qquad (a, b \in \mathcal{A}).$$

We always assume that $\mathbb{F}$ is not of characteristic 2; however, in later sections, $\mathbb{F}$ is usually either $\mathbb{R}$ or $\mathbb{C}$.

The concept of a Jordan algebra was introduced by P. Jordan, J. von Neumann, and E. Wigner [64] to formulate an algebraic model for quantum mechanics. They introduced the notion of an *r-number system* which is, in modern terminology, a finite-dimensional, formally real Jordan algebra. In fact, the term *Jordan algebra* first appeared in an article by A. A. Albert [3]. It denotes an algebra of linear transformations closed in the product

$$A \cdot B = \frac{1}{2}(AB + BA).$$

Although Jordan algebras were motivated by quantum formalism, unexpected and important applications in algebra, geometry and analysis have been discovered. Some of these discoveries are the subject of discussions in ensuing chapters.

On any associative algebra $\mathcal{A}$, a product $\circ$ can be defined by

$$a \circ b = \frac{1}{2}(ab + ba) \qquad (a, b \in \mathcal{A}),$$

where the product on the right-hand side is the original product of $\mathcal{A}$. The algebra $\mathcal{A}$ becomes a Jordan algebra with the product $\circ$. We call this product *special*. A Jordan algebra is called *special* if it is isomorphic to, and hence identified with, a Jordan subalgebra of an associative algebra $\mathcal{A}$ with respect to the special Jordan product $\circ$. Otherwise, it is called *exceptional*.

It is often convenient to express the Jordan identity as an operator identity. Given an algebra $\mathcal{A}$ and $a \in \mathcal{A}$, we define a linear map $L_a : \mathcal{A} \longrightarrow \mathcal{A}$, called *left multiplication by a*, as follows:

$$L_a(x) = ax \qquad (x \in \mathcal{A}).$$

The Jordan identity can be expressed as

$$[L_a, L_{a^2}] = 0 \qquad (a \in \mathcal{A}),$$

where $[\cdot, \cdot]$ is the usual Lie bracket product of linear maps. Given $a, b \in \mathcal{A}$, we define the *quadratic operator* $Q_a : \mathcal{A} \longrightarrow \mathcal{A}$ and *box operator* $a \,\square\, b : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$Q_a = 2L_a^2 - L_{a^2}, \quad a \,\square\, b = L_{ab} + [L_a, L_b]. \tag{1.1}$$

These operators are fundamental in Jordan theory, as is the linearization of the quadratic operator:

$$Q_{a,b} = Q_{a+b} - Q_a - Q_b.$$

Let $\mathcal{A}$ be an algebra and let $a \in \mathcal{A}$. We define $a^0 = \mathbf{1}$ if $\mathcal{A}$ is unital,

$$a^1 = a, \quad a^{n+1} = aa^n \qquad (n = 1, 2, \ldots).$$

The following power associative property depends on the assumption that the scalar field $\mathbb{F}$ for $\mathcal{A}$ is not of characteristic 2.

Theorem 1.1.1 *A Jordan algebra $\mathcal{A}$ is power associative; that is,*

$$a^m a^n = a^{m+n} \qquad (a \in \mathcal{A}; m, n = 1, 2, \ldots).$$

In fact, we have $[L_{a^m}, L_{a^n}] = 0$.

Proof For any α, β in the underlying field $\mathbb{F}$, we have

$$[L_{a+\alpha b+\beta c}, L_{(a+\alpha b+\beta c)^2}] = 0$$

for all $a, b, c \in \mathcal{A}$. Expanding the product, we find that the coefficient of the term $\alpha\beta$ is

$$2[L_a, L_{bc}] + 2[L_b, L_{ca}] + 2[L_c, L_{ab}],$$

which must be 0. Since $\mathbb{F}$ is not of characteristic 2, we have

$$[L_a, L_{bc}] + [L_b, L_{ca}] + [L_c, L_{ab}] = 0.$$

Applying this operator identity to an element $x \in \mathcal{A}$ and using commutativity of the Jordan product yields

$$\begin{aligned}
&(L_a L_{bc} + L_b L_{ca} + L_c L_{ab})(x) \\
&\quad = (L_{bc} L_a + L_{ca} L_b + L_{ab} L_c)(x) \\
&\quad = L_{bc} L_x(a) + L_{bx} L_c(a) + L_{xc} L_b(a) \\
&\quad = (L_{bc} L_x + L_{cx} L_b + L_{xb} L_c)(a) \\
&\quad = (L_x L_{bc} + L_b L_{cx} + L_c L_{xb})(a) \\
&\quad = (L_{((bc)a)} + L_b L_a L_c + L_c L_a L_b)(x).
\end{aligned}$$

Putting $b = a^n$ and $c = a$ in this identity, we obtain a recursive formula,

$$L_{a^{n+2}} = 2L_a L_{a^{n+1}} + L_{a^n} L_{a^2} - L_{a^n} L_a^2, -L_a^2 L_{a^n},$$

which implies that each L_{a^n} is a polynomial in L_a and L_{a^2} which commute. It follows that L_{a^n} commutes with L_{a^m} for all $m, n \in \mathbb{N}$. In particular, we have

$$L_{a^n} L_a(a^m) = L_a L_{a^n}(a^m),$$

and power associativity follows from induction. □

Corollary 1.1.2 *Let $\mathcal{A}$ be a Jordan algebra and let $a \in \mathcal{A}$. The subalgebra $\mathcal{A}(a)$ generated by a in $\mathcal{A}$ is associative.*

In fact, we have the following deeper result. It can be derived from Macdonald's theorem, which states that if an identity in 3 variables is linear in 1 variable and holds in all special Jordan algebras, then it holds in all Jordan algebras. We omit the proof, which can be found, for instance, in the books by Jacobson [62], McCrimmon [88], and Zhevlakov *et al.* [123]. We remark, however, that for Jordan algebras over a field of characteristic 2, a Jordan algebra with a single generator need not be special.

Shirshov–Cohn Theorem *Let $\mathcal{A}$ be a Jordan algebra and let $a, b \in \mathcal{A}$. Then the Jordan subalgebra $\mathcal{B}$ generated by a, b (and* **1***, if $\mathcal{A}$ is unital) is special.*

One can use the Shirshov–Cohn theorem to establish various identities in Jordan algebras. For instance, in any Jordan algebra $\mathcal{A}$, we have the identity

$$2L_a^3 - 3L_{a^2}L_a + L_{a^3} = 0 \tag{1.2}$$

for each $a \in \mathcal{A}$. In other words, we have

$$2a(a(ab)) - 3a^2(ab) + a^3b = 0$$

for $a, b \in \mathcal{A}$. To see this, let $\mathcal{B}$ be the Jordan subalgebra of $\mathcal{A}$ generated by a and b. Then it is special and hence embeds in some associative algebra $(\mathcal{A}', \times)$ with

$$ab = \frac{1}{2}(a \times b + b \times a).$$

In $\mathcal{B}$, we have

$$2a(a(ab)) = \frac{1}{4}(a^3 \times b + 3a^2 \times b \times a + 3a \times b \times a^2 + b \times a^3)$$
$$3a^2(ab) = \frac{1}{4}(3a^3 \times b + 3a^2 \times b \times a + 3a \times b \times a^2 + 3b \times a^3),$$

which, together with $2a^3b = a^3 \times b + b \times a^3$, verifies the identity.

Definition 1.1.3 Two elements a and b in a Jordan algebra $\mathcal{A}$ are said to *operator commute* if the left multiplications L_a and L_b commute. The *centre* of $\mathcal{A}$ is the set $Z(\mathcal{A}) = \{z \in \mathcal{A} : L_z L_a = L_a L_z, \forall a \in \mathcal{A}\}$.

We observe that $L_a L_b = L_b L_a$ if, and only if, $(ax)b = a(xb)$ for all $x \in \mathcal{A}$. Evidently, the centre $Z(\mathcal{A}) = \{z \in \mathcal{A} : (za)b = z(ab), \forall a, b \in \mathcal{A}\}$ is an associative subalgebra of $\mathcal{A}$.

Example 1.1.4 The Cayley algebra $\mathcal{O}$, known as the *octonions*, is a complex nonassociative algebra with a basis $\{e_0, e_1, \ldots, e_7\}$ and satisfies

$$a^2b = a(ab), \qquad ab^2 = (ab)b \qquad (a, b \in \mathcal{O}), \tag{1.3}$$

where e_0 is the identity of $\mathcal{O}$, $e_j^2 = -e_0$ for $j \neq 0$, and the multiplication is encoded in the following Fano plane, consisting of seven points and seven

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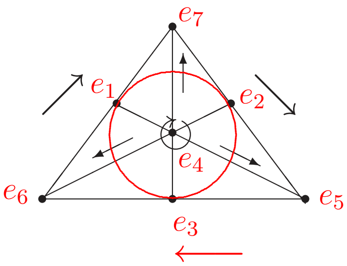
lines. The points are the basis elements but e_0, and the lines are the sides of the triangle, together with the circle. Each line has a cyclic ordering shown by the arrow. If e_i, e_j and e_k are cyclically ordered, then $e_i e_j = -e_j e_i = e_k$. For instance, $e_6 e_2 = (-e_4 e_2) e_2 = -e_4 e_2^2 = e_4$.



Octonion multiplication

The algebra $\mathcal{O}$ is an *alternative algebra* in the sense that the *associator*

$$[x, y, z] = (xy)z - x(yz)$$

is an alternating function of x, y, z: exchanging any two variables entails a sign change of the function. This condition is a reformulation of the multiplication rules in (1.3).

We will denote by $\mathbb{O}$ the real Caylay algebra, which is the real subalgebra of $\mathcal{O}$ with basis $\{e_0, \ldots, e_7\}$. Historically, octonions were discovered by a process of duplicating the real numbers $\mathbb{R}$. Indeed, the complex numbers arise from $\mathbb{R}$ as the product $\mathbb{R} \times \mathbb{R}$ with the multiplication

$$(a, b)(c, d) = (ac - db, bc + da) \qquad (a, b, c, d \in \mathbb{R}).$$

The real associative *quaternion* algebra $\mathbb{H}$ can be constructed by an analogous duplication process. One can define $\mathbb{H}$ as $\mathbb{C} \times \mathbb{C}$ with the multiplication

$$(a, b)(c, d) = (ac - \bar{d}b, b\bar{c} + da) \qquad (a, b, c, d \in \mathbb{C}),$$

which is isomorphic to the following real non-commutative algebra of 2×2 matrices:

$$\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : (a, b) \in \mathbb{C} \times \mathbb{C} \right\}. \tag{1.4}$$

In the identification with this algebra, $\mathbb{H}$ has a basis

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

satisfying

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}, \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

Likewise, $\mathbb{O}$ can be defined as the product $\mathbb{H} \times \mathbb{H}$ with the multiplication

$$(a, b)(c, d) = (ac - \bar{d}b, b\bar{c} + da) \qquad (a, b, c, d \in \mathbb{H}),$$

where the *conjugate* $\bar{c}$ of a quaternion $c = \alpha\mathbf{1} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is defined by

$$\bar{c} = \alpha\mathbf{1} - x\mathbf{i} - y\mathbf{j} - z\mathbf{k},$$

so that the *real part* of c is $\operatorname{Re} c = \frac{1}{2}(c + \bar{c}) = \alpha\mathbf{1}$. A *positive* quaternion is one of the form $\alpha\mathbf{1}$ for some $\alpha > 0$. The basis elements of $\mathbb{H} \times \mathbb{H}$ are

$$\begin{aligned} e_0 &= (\mathbf{1}, 0), & e_1 &= (\mathbf{i}, 0), & e_2 &= (\mathbf{j}, 0), & e_3 &= (\mathbf{k}, 0),\\ e_4 &= (0, \mathbf{1}), & e_5 &= (0, \mathbf{i}), & e_6 &= (0, \mathbf{j}), & e_7 &= (0, \mathbf{k}). \end{aligned}$$

The algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are *quadratic*; that is, each element x satisfies the equation $x^2 = \alpha x + \beta\mathbf{1}$ for some $\alpha, \beta \in \mathbb{R}$, where $\mathbf{1}$ denotes the identity of the algebra. If $x = (a_1, a_2) \in \mathbb{H} \times \mathbb{H}$ with

$$a_n = \alpha_n\mathbf{1} + x_n\mathbf{i} + y_n\mathbf{j} + z_n\mathbf{k} \qquad (n = 1, 2),$$

then we have

$$x^2 = 2\alpha_1 x - (x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2)e_0.$$

Example 1.1.5 A well-known example in Albert [2] of an exceptional Jordan algebra is the 27-dimensional real algebra

$$H_3(\mathbb{O}) = \{(a_{ij})_{1\le i,j\le 3} : (a_{ij}) = (\widetilde{a}_{ji}), a_{ij} \in \mathbb{O}\}$$

of 3×3 matrices over $\mathbb{O}$, Hermitian with respect to the usual involution $\sim$ in $\mathbb{O}$ defined by

$$(\alpha_0 e_0 + \cdots + \alpha_7 e_7)\widetilde{\ } = \alpha_0 e_0 - \cdots - \alpha_7 e_7.$$

The Jordan product is given by

$$A \circ B = \frac{1}{2}(AB + BA) \qquad (A, B \in H_3(\mathbb{O})),$$

where the multiplication on the right is the usual matrix multiplication. We refer to Jacobson [62] and McCrimmon [88] for a more detailed analysis of $H_3(\mathbb{O})$. The exceptionality of $H_3(\mathbb{O})$ involves the so-called *s-identities*, which are valid in all *special* Jordan algebras but not all Jordan algebras. One such identity was first found by Glennie [43]:

$$\begin{aligned}&2Q_x(z)Q_{y,x}Q_z(y^2) - Q_xQ_zQ_{x,y}Q_y(z)\\ &\quad = 2Q_y(z)Q_{x,y}Q_z(x^2) - Q_yQ_zQ_{y,x}Q_x(z),\end{aligned}$$

which does not hold in $H_3(\mathbb{O})$. An alternative proof of exceptionality bypassing s-identities can be found in Hanche-Olsen and Størmer [47].

Definition 1.1.6 An element e in an algebra $\mathcal{A}$ is called an *idempotent* if $e^2 = e$. Two idempotents e and u are said to be *orthogonal* if $eu = ue = 0$. An element $a \in \mathcal{A}$ is called *nilpotent* if $a^n = 0$ for some positive integer n.

Lemma 1.1.7 *Let $\mathcal{A}$ be a unital Jordan algebra with an idempotent e. Let $a \in \mathcal{A}$. The following conditions are equivalent:*

(i) *a and e operator commute.*
(ii) *$Q_e(a) = L_e a$.*
(iii) *a and e generate an associative subalgebra of $\mathcal{A}$.*

Proof (i) $\Rightarrow$ (ii). We have

$$Q_e(a) = 2(L_e^2 - L_e)(a) = 2e(ea) - ea = 2e^2a - ea = ea.$$

(ii) $\Rightarrow$ (iii). Let $\mathcal{B}$ be the subalgebra generated by a and e. By the Shirshov–Cohn theorem, $\mathcal{B}$ is isomorphic to a Jordan subalgebra $\mathcal{B}'$ of an associative algebra $(\mathcal{A}', \times)$ with respect to the special Jordan product. Identify a and e as elements in $\mathcal{B}'$. Then

$$L_e a = \frac{1}{2}(e \times a + a \times e) = Q_e(a) = e \times a \times e,$$

since $e = e^2 = e \times e$. Multiplying the above identity on the left by e, we get $e \times a = e \times a \times e$. Multiplying the identity on the right by e gives $a \times e = e \times a \times e$. Hence $e \times a = a \times e$ and $ea = e \times a$. Hence $(\mathcal{B}', \times)$ is a commutative subalgebra of $(\mathcal{A}, \times)$ and the special Jordan product in $\mathcal{B}'$ is just the product $\times$ and is, in particular, associative.

(iii) $\Rightarrow$ (i). In the proof of Theorem 1.1.1, we have the operator identity

$$[L_e, L_{bc}] + [L_b, L_{ce}] + [L_c, L_{eb}] = 0$$

for all $b, c \in \mathcal{A}$. Putting $c = e$, we have

$$[L_e, L_{be}] + [L_b, L_e] + [L_e, L_{eb}] = 0,$$

which gives

$$2[L_e, L_{be}] = [L_e, L_b]. \tag{1.5}$$

Since $e^2 = e$, in the special Jordan algebra $\mathcal{A}(a, e, \mathbf{1})$ generated by a, e and $\mathbf{1}$, it can be verified easily that

$$a = Q_e(a) + Q_{\mathbf{1}-e}(a)$$

and $Q_{\mathbf{1}-e}(a)e = 0$, as well as $Q_e(a)e = Q_e(a)$. Substituting $Q_{\mathbf{1}-e}(a)$ for b in (1.5), we get $[L_e, L_{Q_{\mathbf{1}-e}(a)}] = 0$. Putting $b = Q_e(a)$ in (1.5) gives $[L_e, L_{Q_e(a)}] = 0$. It follows that

$$[L_e, L_a] = [L_e, L_{Q_e(a)}] + [L_e, L_{Q_{\mathbf{1}-e}(a)}] = 0.$$

□

Lemma 1.1.8 *Let $\mathcal{A}$ be a finite-dimensional associative algebra containing an element a which is not nilpotent and not an identity. Then $\mathcal{A}$ contains a nonzero idempotent, which is a polynomial in a, without constant term.*

Proof We may assume that $\mathcal{A}$ has an identity $\mathbf{1}$. Finite dimensionality implies that there is a nonzero polynomial p of least degree and without constant term, such that $p(a) = 0$. Write $p(x) = x^k q(x)$, where $k \geq 1$ and q is a polynomial, such that $q(0) \neq 0$. The degree $\deg q$ of q is strictly positive, since a is not nilpotent. There are then polynomials q_1 and q_2 with $\deg q_1 < \deg q$ and

$$x^k q_1(x) + q_2(x) q(x) = \mathbf{1},$$

where the nonzero polynomial $g(x) = x^k q_1(x)$ has no constant term and $\deg g < \deg p$. Hence $e = g(a) \neq 0$. We have $e^2 = e$, since $a^{2k} q_1(a) + a^k q_2(a) q(a) = a^k$ and

$$g(a)^2 - g(a) = a^{2k} q_1(a)^2 - a^k q_1(a) = a^k q_2(a) q(a) q_1(a) = 0.$$

□

Lemma 1.1.9 *Let $\mathcal{A}$ be a Jordan algebra. Then an element $a \in \mathcal{A}$ is nilpotent if and only if the left multiplication $L_a : \mathcal{A} \longrightarrow \mathcal{A}$ is nilpotent.*

Proof If L_a is nilpotent, then $a^{n+1} = L_a^n(a)$ implies that a is nilpotent. Conversely, for any $a \in \mathcal{A}$ with $a^n = 0$, we show that L_a is nilpotent by induction on the exponent n. The assertion is trivially true if $n = 1$. Given that the assertion is true for n, we consider $a^{n+1} = 0$. We have $(a^2)^n = 0 = (a^3)^n$, and therefore L_{a^2} and L_{a^3} are nilpotent, by the inductive hypothesis. It follows from the identity

$$2L_a^3 = 3L_{a^2} L_a - L_{a^3}$$

that L_a is nilpotent. □

Given an idempotent e in a Jordan algebra $\mathcal{A}$, the left multiplication $L_e : \mathcal{A} \longrightarrow \mathcal{A}$ satisfies the equation

$$2L_e^3 - 3L_e^2 + L_e = 0 \tag{1.6}$$

by the identity (1.2). Hence an eigenvalue α of L_e is a root of

$$2\alpha^3 - 3\alpha^2 + \alpha = 0$$

and is 0, $\frac{1}{2}$ or 1. If $\mathcal{A}$ is associative, then $L_e^2 = L_e$ and $\frac{1}{2}$ is not an eigenvalue of L_e. Nevertheless, we denote the eigenspaces of $2L_e$ by

$$\mathcal{A}_k(e) = \{x \in \mathcal{A} : 2ex = kx\} \qquad (k = 0,\ 1,\ 2)$$

and call $\mathcal{A}_k(e)$ the *Peirce k-space* of e. The earlier remark implies that $\mathcal{A}_1(e) = \{0\}$ if $\mathcal{A}$ is associative.

We define two linear operators, $Q_e : \mathcal{A} \longrightarrow \mathcal{A}$, and $Q_e^\perp : \mathcal{A} \longrightarrow \mathcal{A}$, by

$$Q_e = 2L_e^2 - L_e, \quad Q_e^\perp = 4(L_e - L_e^2). \tag{1.7}$$

Evidently, L_e commutes with both Q_e and $Q_e^\perp$. Using the equation (1.6), one can easily establish

$$L_e Q_e = Q_e = Q_e^2, \quad L_e Q_e^\perp = \frac{1}{2} Q_e^\perp = \frac{1}{2}(Q_e^\perp)^2, \quad L_e(I - Q_e - Q_e^\perp) = 0,$$

where I is the identity operator on $\mathcal{A}$ and Q_e and $Q_e^\perp$ are mutually orthogonal. It follows that

$$\mathcal{A}_2(e) = Q_e(\mathcal{A}), \quad \mathcal{A}_1(e) = Q_e^\perp(\mathcal{A}), \quad \mathcal{A}_0(e) = (I - Q_e - Q_e^\perp)(\mathcal{A}), \tag{1.8}$$

which gives rise to the following *Peirce decomposition* of $\mathcal{A}$:

$$\mathcal{A} = \mathcal{A}_0(e) \oplus \mathcal{A}_1(e) \oplus \mathcal{A}_2(e).$$

We will return to the Peirce decomposition with more details in the more general setting of Jordan triple systems. We note for the time being that the Peirce spaces $\mathcal{A}_0(e)$ and $\mathcal{A}_2(e)$ are Jordan subalgebras of $\mathcal{A}$, as shown below. We also note that $\mathcal{A}_2(e)$ never vanishes.

Lemma 1.1.10 *The Peirce spaces of an idempotent e in a Jordan algebra $\mathcal{A}$ satisfy*

$$\mathcal{A}_0(e)\mathcal{A}_0(e) \subset \mathcal{A}_0(e), \quad \mathcal{A}_1(e)\mathcal{A}_1(e) \subset \mathcal{A}_0(e) \oplus \mathcal{A}_2(e), \quad \mathcal{A}_2(e)\mathcal{A}_2(e) \subset \mathcal{A}_2(e).$$

Proof We first prove the second inclusion. Let $x, y \in \mathcal{A}_1(e)$ and let $xy = a_0 + a_1 + a_2$ be the Peirce decomposition of xy. We have

$$\begin{aligned} 0 &= [L_y, L_{ex}] + [L_e, L_{xy}] + [L_x, L_{ye}] \\ &= \frac{1}{2}[L_y, L_x] + [L_e, L_{xy}] + \frac{1}{2}[L_x, L_y] \\ &= [L_e, L_{xy}]. \end{aligned}$$

In particular, $[L_e, L_{xy}](e) = 0$ gives

$$\begin{aligned} 0 &= e(xy) - e(e(xy)) \\ &= a_2 + \frac{1}{2}a_1 - e\left(a_2 + \frac{1}{2}a_1\right) \\ &= a_2 + \frac{1}{2}a_1 - a_2 - \frac{1}{4}a_1 = \frac{1}{4}a_1. \end{aligned}$$

Hence $xy \in \mathcal{A}_0(e) \oplus \mathcal{A}_2(e)$.

Let $x, y \in \mathcal{A}_j(e)$, where $j = 0, 2$. Then we have $[L_e, L_{x^2}] = 0$ by the first equation earlier in the proof. Using this and expanding $((x+e)y)(x+e)^2 = (x+e)(y(x+e)^2)$, we obtain

$$2(xy)(xe) + (xy)e + 2(ey)(xe) = 2x(y(xe)) + x(ye) + 2(ey)(xe),$$

which gives $(xy)e = \frac{j}{2}xy$. This proves the first and the last inclusion. □

Definition 1.1.11 An idempotent e in a Jordan algebra $\mathcal{A}$ is called *maximal* if the Peirce 0-space $\mathcal{A}_0(e)$ is $\{0\}$. A nonzero idempotent e is called *primitive* if there are no nonzero orthogonal idempotents u and v satisfying $e = u + v$.

Lemma 1.1.12 *Let $\mathcal{A}$ be a finite-dimensional Jordan algebra which contains no nonzero nilpotent element. Then $\mathcal{A}$ contains a maximal idempotent.*

Proof Ignore the trivial case $\mathcal{A} = \{0\}$. Applying Lemma 1.1.8 to an associative subalgebra of $\mathcal{A}$ generated by a nonzero element, one finds a nonzero idempotent e. If $\mathcal{A}_0(e) \neq \{0\}$, then again one can pick a nonzero idempotent $u \in \mathcal{A}_0(e)$. Then $e' = e + u$ is an idempotent and $\mathcal{A}_0(e) \subset \mathcal{A}_0(e')$. Since $u \in \mathcal{A}_0(e') \backslash \mathcal{A}_0(e)$, we have $\dim \mathcal{A}_0(e) < \dim \mathcal{A}_0(e')$. By finite dimensionality of $\mathcal{A}$, this process of increasing dimension must stop, yielding a maximal idempotent. □

Proposition 1.1.13 *Let $\mathcal{A}$ be a finite-dimensional Jordan algebra which contains no nonzero nilpotent element. Then $\mathcal{A}$ has an identity.*

Proof By Lemma 1.1.12, $\mathcal{A}$ contains a maximal idempotent e such that

$$\mathcal{A} = \mathcal{A}_1(e) \oplus \mathcal{A}_2(e).$$