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for Susie

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Preface

Group cohomology reveals a deep relation between algebra and topology. A group determines a topological space in a natural way, its classifying space. The cohomology ring of a group is defined to be the cohomology ring of its classifying space. The challenges are to understand how the algebraic properties of a group are related to its cohomology ring, and to compute the cohomology rings of particular groups.

A fundamental fact is that the cohomology ring of any finite group is finitely generated. So there is some finite description of the whole cohomology ring of a finite group, but it is not clear how to find it. A central problem in group cohomology is to find an upper bound for the degrees of generators and relations for the cohomology ring. If we can do that, then there are algorithms to compute the cohomology in low degrees and therefore compute the whole cohomology ring.

Peter Symonds made a spectacular advance in 2010: for any finite group G with a faithful complex representation of dimension n at least 2 and any prime number p, the mod p cohomology ring of G is generated by elements of degree at most n^2 [130]. Not only is this the first known bound for generators of the cohomology ring; it is also nearly an optimal bound among arbitrary finite groups, as we will see.

This book proves Symonds's theorem (Corollary 4.3) and several new variants and improvements of it. Some involve algebro-geometric analogs of the cohomology ring. Namely, Morel-Voevodsky and I independently showed how to view the classifying space of an algebraic group G (e.g., a finite group) as a limit of algebraic varieties in a natural way. That allows the definition of the Chow ring of algebraic cycles on the classifying space BG [107, proposition 2.6]; [138]. A major goal of algebraic geometry is to compute the Chow ring for varieties of interest, since that says something meaningful about all subvarieties of the variety.

The fact that not all the cohomology of BG is represented by algebraic cycles (even for abelian groups G) is the source of Atiyah-Hirzebruch's counterexamples to the integral Hodge conjecture [8, 137, 138]. It is a natural problem of "motivic homotopy theory" to understand the Chow ring and more generally the motivic cohomology of classifying spaces BG. Concretely, computing the Chow ring of BG amounts to computing the Chow groups of the quotients by G of all representations of G. Such quotients are extremely special among all varieties, but they have been fundamental examples in algebraic geometry for more than 150 years. Computing their Chow groups is a fascinating problem. (Rationally, the calculations are easy; the interest is in integral or mod p calculations.)

Bloch generalized Chow groups to a bigraded family of groups, now called motivic cohomology. A great achievement of motivic homotopy theory is the proof by Voevodsky and Rost of the Bloch-Kato conjecture [145, theorem 6.16]. A corollary, the Beilinson-Lichtenbaum conjecture (Theorem 6.9), says that for any smooth variety over a field, a large range of motivic cohomology groups with finite coefficients map isomorphically to etale cohomology. Etale cohomology is a more computable theory, which coincides with ordinary cohomology in the case of complex varieties. Thus the Beilinson-Lichtenbaum conjecture is a powerful link between algebraic geometry and topology.

Chow groups are the motivic cohomology groups of most geometric interest, but they are also farthest from the motivic cohomology groups that are computed by the Beilinson-Lichtenbaum conjecture. A fundamental difficulty in computing Chow groups is "etale descent": for a finite Galois etale morphism $X \to Y$ of schemes, how are the Chow groups of X and Y related? This is easy after tensoring with the rationals; the hard case of etale descent is to compute Chow groups integrally, or with finite coefficients. Etale descent is well understood for etale cohomology, and hence for many motivic cohomology groups with finite coefficients.

The problem of etale descent provides some motivation for trying to compute the Chow ring of classifying spaces of finite groups G. Computing the Chow ring of BG means computing the Chow ring of certain varieties Y which have a covering map $X \to Y$ with Galois group G (an approximation to $EG \to BG$) such that X has trivial Chow groups. Thus the Chow ring of BG is a model case in seeking to understand etale descent for Chow groups.

Chow rings can be generalized in various ways, for example, to algebraic cobordism and motivic cohomology. Another direction of generalization leads to unramified cohomology, cohomological invariants of algebraic groups [47], and obstructions to rationality for quotient varieties [17, 76]. All of these invariants are worth computing for classifying spaces, but we largely focus on the most classical case of Chow rings. Some of our methods will certainly be useful for these more general invariants. For example, finding generators for the

Chow ring of a smooth variety automatically gives generators of its algebraic cobordism, by Levine and Morel [96, theorem 1.2.19].

This book mixes algebraic geometry and algebraic topology, and few readers will have all the relevant background. With that in mind, I include brief introductions to several of the theories we use. Chapter 1 introduces group cohomology. Chapter 2 summarizes the basic properties of the Chow ring of a smooth variety without proof, and then introduces equivariant Chow rings in more detail, including some calculations. I hope this allows topologists who have seen a little algebraic geometry to get some feeling for Chow rings. However, large parts of the book are devoted to group cohomology, including many new results, and topologists may prefer to concentrate on those parts.

An explicit bound for the degrees of generators of the Chow ring of BG, of the same form as Symonds's bound for cohomology, was given in 1999 [138, theorem 14.1]. The first new result of this book is to improve the earlier bound for the Chow ring by about a factor of two: for any finite group G with a faithful complex representation of dimension n at least 3, the Chow ring of BG is generated by elements of degree at most $n(n-1)/2$. Moreover, this improved bound is optimal, for all n (Chapter 5).

For a p-group, Chapter 7 gives a stronger bound for the degrees of generators of the cohomology ring and the Chow ring. For the cohomology ring of a p-group, this result goes well beyond Symonds's general bound. The case of p-groups is central in the cohomology theory of finite groups, with many questions reducing to that case. It may be that these bounds for p-groups can be improved further.

Chapter 8 proves some of the fundamental theorems on the cohomology and Chow ring of a finite group. First, there is Quillen's theorem that, up to F-isomorphism (loosely, "up to pth powers"), the cohomology ring of a finite group is determined by the inclusions among its elementary abelian subgroups. We prove Yagita's theorem that the Chow ring of a finite group, up to F-isomorphism, has the same description in terms of the elementary abelian subgroups. It follows that the cycle map from the Chow ring of a finite group to the cohomology ring is an F-isomorphism.

Next, we give a strong bound for the degrees of generators of the Chow ring of a finite group modulo transfers from proper subgroups. In particular, for a group with a faithful representation of dimension n and any prime number p, the mod p Chow ring is generated by elements of degree less than n modulo transfers from proper subgroups (Corollary 10.5). (In fact, we only need to consider transfers from a particular class of subgroups, centralizers of elementary abelian p-subgroups.) This result reduces the problem of finding generators for the Chow ring of a given group to the problem of finding generators for the Chow groups of certain low-dimensional quotient varieties. Symonds proved

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the analogous very strong bound for the cohomology ring of a finite group modulo transfers from proper subgroups, and we give a version of his argument (Corollary 10.3).

In examples, the Chow ring of a finite group G always turns out to be simpler than the cohomology ring, and it seems to be closely related to the complex representation theory of G. In that direction, I conjectured that the Chow ring of any finite group was generated by transfers of Euler classes (top Chern classes) of complex representations [138]. That was disproved by Guillot for a certain group of order 2^7, the extraspecial 2-group 2^{1+6}_+ [62]. It would be good to find similar examples at odd primes. Nonetheless, the theorem on the Chow ring modulo transfers gives a class of p-groups for which the question has a positive answer. Namely, the Chow ring of a p-group with a faithful complex representation of dimension at most $p+2$ consists of transferred Euler classes (Theorem 11.1). This includes all 2-groups of order at most 32, and all p-groups of order at most p^4 with p odd.

We extend Symonds's theorem on the Castelnuovo-Mumford regularity of the cohomology ring to the Chow ring of the classifying space of a finite group (Theorem 6.5). We also bound the regularity of motivic cohomology (Theorem 6.10). It follows, for example, that all our bounds on generators for the Chow ring also lead to bounds on the relations. In each case, our upper bound for the degree of the relations is twice the bound for the degree of the generators. Another application is an identification of the motivic cohomology of a classifying space BG in high weights with the ordinary (or etale) cohomology. This statement goes beyond the range where motivic cohomology and etale cohomology are the same for arbitrary varieties, as described by the Beilinson-Lichtenbaum conjecture.

Let G be a finite group with a faithful complex representation of dimension n. Chapter 12 shows that the cohomology of G is determined by the cohomology of certain subgroups (centralizers of elementary abelian subgroups) in degrees less than $2n$. This was conjectured by Kuhn, who was continuing a powerful approach to group cohomology developed by Henn, Lannes, and Schwartz [86, 69]. We also prove an analogous result for the Chow ring: the Chow ring of a finite group is determined by the cohomology of centralizers of elementary abelian subgroups in degrees less than n. This is a strong computational tool, in a slightly different direction from the bounds for degrees of generators. The proof is inspired by Kuhn's ideas on group cohomology.

For a finite group G, Henn, Lannes, and Schwartz found that much of the complexity of the cohomology ring of G is described by one number, the "topological nilpotence degree" d_0 of the cohomology ring. This number is defined in terms of the cohomology ring as a module over the Steenrod algebra, but it is also equal to the optimal bound for determining the cohomology of G in terms of the low-degree cohomology of centralizers of elementary abelian subgroups.

Section 13.5 gives the first calculations of the topological nilpotence degree d_0 for some small p-groups, such as the groups of order p^3. In these examples, d_0 turns out to be much smaller than known results would predict. Improved bounds for d_0 would be a powerful computational tool in group cohomology.

To understand the cohomology of finite groups, it is important to compute the cohomology of large classes of p-groups. The cohomology of particular finite groups such as the symmetric groups and the general linear groups over finite fields F (with coefficients in $\mathbf{F}_p$ for p invertible in F) were computed many years ago by Nakaoka and Quillen. The calculations were possible because the Sylow p-subgroups of these groups are very special (iterated wreath products). To test conjectures in group cohomology, it has been essential to make more systematic calculations for p-groups, such as Carlson's calculation of the cohomology of all 267 groups of order 2^6 [26, appendix]. More recently, Green and King computed the cohomology of all 2328 groups of order 2^7 and all 15 groups of order 3^4 or 5^4 [51, 52]. In that spirit, we begin the systematic calculation of Chow rings of p-groups. Chapter 13 computes the Chow rings of all 5 groups of order p^3 and all 14 groups of order 16. Chapter 14 computes the Chow ring for all 15 groups of order $3^4 = 81$, and for 13 of the 15 groups of order p^4 with $p \geq 5$. Most of the proofs use only Chow rings, but the hardest cases also use calculations of group cohomology by Leary and Yagita.

One tantalizing example for which the Chow ring is not yet known is the group G of strictly upper triangular matrices in $GL(4, \mathbf{F}_p)$, which has order p^6. The machinery in this book should at least make that calculation easier. For p odd, Kriz and Lee showed that the Morava K-theory $K(2)^*BG$ is not concentrated in even degrees, disproving a conjecture of Hopkins, Kuhn, and Ravenel [83, 84]. It seems to be unknown whether the complex cobordism of BG is concentrated in even degrees in this example. Until this is resolved, it remains a possibility that the Chow ring of BG may map isomorphically to the quotient $MU^*(BG) \otimes_{MU^*} \mathbf{Z}$ of complex cobordism for every complex algebraic group G (including finite groups), as conjectured in [138]. Yagita strengthened this conjecture to say that algebraic cobordism Ω^*BG should map isomorphically to the topologically defined MU^*BG for every complex algebraic group G [154, conjecture 12.2].

Chapter 15 gives examples of p-groups for any prime number p such that the geometric and topological filtrations on the complex representation ring are different. When $p = 2$, Yagita has also given such examples [156, corollary 5.7]. A representation of G determines a vector bundle on BG, and these two filtrations describe the "codimension of support" of a virtual representation in the algebro-geometric or the topological sense. Atiyah conjectured that the (algebraically defined) γ-filtration of the representation ring was equal to the topological filtration [6], but that was disproved by Weiss, Thomas, and (for p-groups) Leary and Yagita [93]. Since the geometric filtration lies between the

γ and topological filtrations, the statement that the geometric and topological filtrations can be different is stronger. The examples use Vistoli's calculation of the Chow ring of the classifying space of $PGL(p)$ for prime numbers p [143].

Chapter 16 constructs an Eilenberg-Moore spectral sequence in motivic cohomology for schemes with an action of a split reductive group. The spectral sequence was defined by Krishna with rational coefficients [82, theorem 1.1]. We give an integral result, as far as possible. The Eilenberg-Moore spectral sequence in ordinary cohomology is a basic tool in homotopy theory. Given the cohomology of the base and total space of a fibration, the spectral sequence converges to the cohomology of a fiber. The reason for including the motivic Eilenberg-Moore spectral sequence in this book is to clarify the relation between the classifying space of an algebraic group and its finite-dimensional approximations.

Finally, Chapter 17 considers the Chow Künneth conjecture: for a finite group G and a field k containing enough roots of unity, the natural map $CH^*BG_k \otimes_{\mathbf{Z}} CH^*X \to CH^*(BG_k \times X)$ should be an isomorphism for all smooth schemes X over k. This would in particular imply that the Chow ring of BG_K is the same for all field extensions K of k. Although there is no clear reason to believe the conjecture, we prove some partial results for arbitrary groups, and prove the second version of the conjecture completely for p-groups with a faithful representation of dimension at most $p + 2$. Chapter 18 is a short list of open problems. The Appendix tabulates several invariants of the Chow rings of p-groups of order at most p^4.

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