1

Group Cohomology

This chapter gives the topological and algebraic definitions of group cohomology. We also define equivariant cohomology.

Although we give the basic definitions, a beginner may have to refer to other sources. Brown [24] is an excellent introduction to group cohomology. Group cohomology is also treated in general texts on homological algebra such as Weibel [149]. Some of the main advanced books on the cohomology of finite groups are Adem-Milgram [1], Benson [12], and Carlson [26].

Group cohomology unified many earlier ideas in algebra and topology. It was defined in 1943–1945 by Eilenberg and MacLane, Hopf and Eckmann, and Freudenthal.

1.1 Definition of group cohomology

Group cohomology arises from the fact that any group determines a topological space, as follows. Let G be a topological group. The special case where G is a discrete group is a rich subject in itself. Say that G acts *freely* on a space X if the map $G \times X \to X \times X$, $(g, x) \mapsto (x, gx)$, is a homeomorphism from $G \times X$ onto its image. By Serre, if a Lie group G acts freely on a metrizable topological space X, then the map $X \to X/G$ is a principal G-bundle, meaning that it is locally a product $U \times G \to U$ [109, section 4.1].

There is always a contractible space EG on which G acts freely. The *classifying space* of G is the quotient space of EG by the action of G, $BG = EG/G$. Any two classifying spaces for G that are paracompact are homotopy equivalent [72, definition 4.10.5, exercise 4.9]. If G is a discrete group, a classifying space of G can also be described as a connected space with fundamental group G whose universal cover is contractible, or as an Eilenberg-MacLane space $K(G, 1)$.

The cohomology of the classifying space of a topological group G is well-defined, because the classifying space is unique up to homotopy equivalence. In particular, for any commutative ring R, the cohomology $H^*(BG, R)$ is a graded-commutative R-algebra that depends only on G. For a discrete group G, we call $H^*(BG, R)$ the *cohomology of* G with coefficients in R; confusion should not arise with the cohomology of G as a topological space, which is uninteresting for G discrete. A fundamental challenge is to understand the relation between algebraic properties of a group and algebraic properties of its cohomology ring.

The cohomology of a group G manifestly says something about the cohomology of certain quotient spaces. More generally, for any space X on which G acts freely, there is a fibration

$$X \to (X \times EG)/G \to BG,$$

where the total space is homotopy equivalent to X/G. The resulting spectral sequence $H^*(BG, H^*X) \Rightarrow H^*(X/G)$, defined by Hochschild and Serre, shows that the cohomology of G gives information about the cohomology of any quotient space by G.

Another role of the classifying space of a group G is that it classifies principal G-bundles. By definition, a principal G-bundle over a space X is a space E with a free G-action such that $X = E/G$. The classifying space BG classifies principal G-bundles in the sense that for any CW-complex X, there is a one-to-one correspondence between isomorphism classes of principal G-bundles over X and homotopy classes of maps $X \to BG$. (Explicitly, we have a "universal" G-bundle $EG \to BG$, and a map $f : X \to BG$ defines a G-bundle over X by pulling back: let E be the fiber product $X \times_{BG} EG$.)

Therefore, computing the cohomology of the classifying space gives information about the classification of principal G-bundles over an arbitrary space. Namely, an element $u \in H^i(BG, R)$ gives a *characteristic class* for G-bundles: for any G-bundle E over a space X, we get an element $u(E) \in H^i(X, R)$, by pulling back u via the map $X \to BG$ corresponding to E.

A homomorphism $G \to H$ of topological groups determines a homotopy class of continuous maps $BG \to BH$. For example, we can view this as the obvious map $(EG \times EH)/G \to EH/H = BH$. As a result, given a commutative ring R, a homomorphism $G \to H$ determines a "pullback map" on group cohomology:

$$H^*(BH, R) \to H^*(BG, R)$$

Example The classifying space of the group $\mathbf{Z}/2$ can be viewed as the infinite real projective space $\mathbf{RP}^\infty = \cup_{n \geq 0} \mathbf{RP}^n$. Its cohomology with coefficients in the field $\mathbf{F}_2 = \mathbf{Z}/2$ is a polynomial ring,

$$H^*(B\mathbf{Z}/2, \mathbf{F}_2) = \mathbf{F}_2[x],$$

where x has degree 1. On the finite-dimensional approximation $\mathbf{RP}^n$ to $B\mathbf{Z}/2$, x restricts to the class of a hyperplane $\mathbf{RP}^{n-1} \subset \mathbf{RP}^n$.

Example The classifying space of the general linear group $GL(n, \mathbf{C})$ can be viewed as the Grassmannian $\mathrm{Gr}(n, \infty)$ of n-dimensional complex linear subspaces in $\mathbf{C}^\infty$. The cohomology of this classifying space is a polynomial ring,

$$H^*(BGL(n, \mathbf{C}), \mathbf{Z}) = \mathbf{Z}[c_1, \dots, c_n].$$

A standard reference for this calculation is Milnor-Stasheff [106, theorem 14.5]. (We determine the Chow ring of $BGL(n)$ in Theorem 2.13, by a method that also works for cohomology.) These generators $c_1, \dots, c_n$ are called Chern classes. They have degrees $|c_i| = 2i$, meaning that $c_i \in H^{2i}(BGL(n, \mathbf{C}), \mathbf{Z})$.

There is an equivalence of categories between rank-n complex vector bundles V over a space X and principal $GL(n, \mathbf{C})$-bundles E over X; given E, we define $V = (E \times \mathbf{C}^n)/GL(n, \mathbf{C})$. Therefore, the Chern classes give invariants for complex vector bundles V over any space X, $c_i(V) \in H^{2i}(X, \mathbf{Z})$.

Note that $GL(n, \mathbf{C})$ deformation retracts onto the unitary group $U(n)$. (For a matrix in $GL(n, \mathbf{C})$, the columns form a basis for $\mathbf{C}^n$. The Gram-Schmidt process shows how to move them continuously to an orthonormal basis for $\mathbf{C}^n$, which can be identified with an element of $U(n)$.) It follows that the resulting continuous map $BU(n) \to BGL(n, \mathbf{C})$ is a homotopy equivalence. So the previous calculation can be restated as $H^*(BU(n), \mathbf{Z}) = \mathbf{Z}[c_1, \dots, c_n]$.

As a result, for any compact Lie group G (e.g., a finite group), any complex representation $G \to U(n)$ has Chern classes $c_i \in H^i(BG, \mathbf{Z})$ for $i = 1, \dots, n$, defined by the pullback map $H^*(BU(n), \mathbf{Z}) \to H^*(BG, \mathbf{Z})$. We can also say that a representation of G determines a vector bundle on BG, and these are the Chern classes of that bundle.

Although we won't need this, it is interesting to note that for compact Lie groups G and H, a continuous map $BG \to BH$ need not be homotopic to one coming from a homomorphism $G \to H$. Sullivan gave the first example: for any odd positive integer a, there is an "unstable Adams operation" $\psi^a : BSU(2) \to BSU(2)$ that induces multiplication by a^2 on $H^4(BSU(2), \mathbf{Z}) \cong \mathbf{Z}$ [128, corollaries 5.10, 5.11]. Only the map ψ^1 (the identity map) comes from a group homomorphism $SU(2) \to SU(2)$.

1.2 Equivariant cohomology and basic calculations

Let G be a topological group acting on a topological space X. The *(Borel) equivariant cohomology* of X with respect to G is $H^i_G(X, R) = H^i((X \times EG)/G, R)$. That is, we make the action of G free without changing the homotopy type of X, and then take the quotient by G. In particular, if G acts freely on X, then equivariant cohomology is simply the cohomology of the quotient space, $H^i_G(X) = H^i(X/G)$. At the other extreme, we write H^i_G for the G-equivariant cohomology of a point (with a given coefficient ring, which we usually take to be the field $\mathbf{F}_p = \mathbf{Z}/p$ for a prime number p):

$$H^i_G := H^i_G(\text{point}, \mathbf{F}_p) = H^i(BG, \mathbf{F}_p).$$

Evens and Venkov proved the finite generation of the cohomology ring of a finite group. We give Venkov's elegant proof using equivariant cohomology, which works more generally for compact Lie groups [12, vol. 2, theorem 3.10.1]. Venkov's method helped to inspire Quillen's work on group cohomology and the later developments described in this book.

Theorem 1.1 *Let G be a compact Lie group and R a noetherian ring. Then $H^*(BG, R)$ is a finitely generated R-algebra. For any closed subgroup H of G, $H^*(BH, R)$ is a finitely generated module over $H^*(BG, R)$.*

Here the map $BH \to BG$ gives a ring homomorphism $H^*(BG, R) \to H^*(BH, R)$, and so we can view $H^*(BH, R)$ as a module over $H^*(BG, R)$.

Proof Every compact Lie group G has a faithful complex representation, giving an imbedding of G into $U(n)$ for some n [20, theorem III.4.1]. Since $H^*(BU(n), R) = R[c_1, \ldots, c_n]$ is a finitely generated R-algebra, the first statement of the theorem follows if we can show that $H^*(BG, R)$ is a finitely generated module over the ring of Chern classes $H^*(BU(n), R)$. (This will also imply the second statement of the theorem: for $H \subset G \subset U(n)$, $H^*(BH, R)$ is a finitely generated module over $R[c_1, \ldots, c_n]$ and hence over $H^*(BG, R)$.)

The Leray-Serre spectral sequence of the fibration $U(n)/G \to BG \to BU(n)$ has the form

$$E_2^{ij} = H^i(BU(n), H^j(U(n)/G, R)) \Rightarrow H^{i+j}(BG, R).$$

Since $U(n)/G$ is a closed manifold, its cohomology groups are finitely generated and are zero in degrees greater than the dimension of $U(n)/G$ (which is $n^2 - \dim(G)$). So the E_2 term of the spectral sequence has finitely many rows, each of which is a finitely generated module over $H^*(BU(n), R)$. Since the ring $H^*(BU(n), R)$ is noetherian, every submodule of a finitely generated module over $H^*(BU(n), R)$ is finitely generated, and hence any quotient of a submodule is finitely generated. The differentials in the spectral sequence are linear over $H^*(BU(n), R)$, and so the E_∞ term of the spectral sequence also has finitely many rows, each of which is a finitely generated module over $H^*(BU(n), R)$. Since $H^*(BG, R)$ is filtered with these rows as quotients, $H^*(BG, R)$ is a finitely generated module over $H^*(BU(n), R)$. □

The cohomology of abelian groups is easy to compute. To state the result, write $R\langle x_1, \ldots, x_n\rangle$ for the free graded-commutative algebra over a commutative ring R. This is a graded ring, with given degrees $|x_i| \in \mathbf{Z}$ for the generators, which is the tensor product of the polynomial ring on the generators of even degree with the exterior algebra on the generators of odd degree. We use this notation only for rings R containing $1/2$. (The point is that the cohomology ring of a topological space with coefficients in any commutative ring R is graded-commutative in the sense that $xy = (-1)^{|x||y|}yx$, but only when R contains $1/2$ does this imply that $x^2 = 0$ for x of odd degree. The $\mathbf{F}_2$-cohomology ring of a topological space is commutative in the naive sense.)

Theorem 1.2 *The cohomology ring of $B(S^1)^n$ with any coefficient ring R is the polynomial ring $R[y_1, \ldots, y_n]$ with $|y_i| = 2$.*

The integral cohomology ring of $B(\mathbf{Z}/n)$ for a positive integer n is $\mathbf{Z}[y]/(ny)$ with $|y| = 2$. The generator y can be viewed as the first Chern class of a 1-dimensional complex representation $\mathbf{Z}/n \subset U(1)$.

The $\mathbf{F}_2$-cohomology ring of $B(\mathbf{Z}/2)$ is the polynomial ring $\mathbf{F}_2[x]$ with $|x| = 1$. The $\mathbf{F}_2$-cohomology ring of $B(\mathbf{Z}/2^r)$ for $r \geq 2$ is $\mathbf{F}_2[x, y]/(x^2)$ with $|x| = 1$ and $|y| = 2$.

Finally, for an odd prime number p and any $r \geq 1$, the $\mathbf{F}_p$-cohomology ring of $B(\mathbf{Z}/p^r)$ is $\mathbf{F}_p\langle x, y\rangle$ with $|x| = 1$ and $|y| = 2$.

These results can be proved by viewing BS^1 as the infinite projective space $\mathbf{CP}^\infty$ and viewing $B\mathbf{Z}/n$ as the principal S^1-bundle over $\mathbf{CP}^\infty$ whose first Chern class is n times a generator of $H^2(\mathbf{CP}^\infty, \mathbf{Z})$. Or one can give an algebraic proof, as in [1, section II.4]. These results determine the cohomology of BG for any abelian compact Lie group G using the Künneth formula, since $B(G \times H) = BG \times BH$.

For any topological space X, the *Bockstein* $\beta\colon H^i(X, \mathbf{Z}/p) \to H^{i+1}(X, \mathbf{Z})$ is the boundary map associated to the short exact sequence of coefficient groups

$$0 \to \mathbf{Z} \underset{p}{\to} \mathbf{Z} \to \mathbf{Z}/p \to 0.$$

The resulting long exact sequence shows that the Bockstein vanishes on integral classes. The composition $H^i(X, \mathbf{Z}/p) \underset{\beta}{\to} H^{i+1}(X, \mathbf{Z}) \to H^{i+1}(X, \mathbf{Z}/p)$ is also called the Bockstein. Because the Bockstein vanishes on integral classes, $\beta^2 = 0$. The Bockstein is a derivation on the mod p cohomology ring of any space, in the sense that $\beta(xy) = \beta(x)y + (-1)^{|x|}x\,\beta(y)$ for x, y in $H^*(X, \mathbf{Z}/p)$ [68, section 3.E]. We also note that $\beta x = x^2$ for all $x \in H^1(X, \mathbf{Z}/2)$.

The Bockstein on mod p cohomology is a convenient way to encode some information about integral cohomology. For that reason, we record the Bockstein on the mod p cohomology of the cyclic group $\mathbf{Z}/p^r$: in the preceding notation, $\beta y = 0$ since y is an integral class, and βx is equal to y for $r = 1$ (where we write $y = x^2$ for the group $\mathbf{Z}/2$) and to zero for $r > 1$.

1.3 Algebraic definition of group cohomology

We now present the purely algebraic definition of the cohomology of a discrete group. This is good to know, but it is not used in the rest of the book.

The algebraic definition of group cohomology is one answer to the question of how the algebraic structure of a group determines its cohomology ring. It does not answer all the questions. For example, what special properties does the mod p cohomology ring of a finite simple group have? Or a finite p-group?

To give the definition, let G be a discrete group. We can identify modules over the group ring $\mathbf{Z}G$ with abelian groups on which G acts by automorphisms. Consider the functor from $\mathbf{Z}G$-modules M to abelian groups given by the invariants $M^G := \{x \in M : gx = x \text{ for all } g \in G\}$. This is a left exact functor, meaning that a short exact sequence $0 \to A \to B \to C \to 0$ of $\mathbf{Z}G$-modules determines an exact sequence:

$$0 \to A^G \to B^G \to C^G.$$

We can therefore consider the right-derived functors of M^G, which are called the *cohomology* of G with coefficients in M, $H^i(G, M)$. In particular, $H^0(G, M) = M^G$, and a short exact sequence of $\mathbf{Z}G$-modules gives a long exact sequence of cohomology groups:

$$0 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \to H^1(G, A) \to \cdots .$$

Moreover, this notion of group cohomology agrees with the topological definition: for any $\mathbf{Z}G$-module M, $H^*(G, M)$ is isomorphic to the cohomology of the topological space BG with coefficients in the locally constant sheaf associated to M. In particular, if G acts trivially on M, then this is the usual notion of cohomology of the space BG with coefficients in the abelian group M.

We recall how right-derived functors are defined: choose a resolution

$$0 \to M \to I_0 \to I_1 \to \cdots$$

of M by injective $\mathbf{Z}G$-modules and define $H^*(G, M)$ to be the cohomology of the chain complex:

$$0 \to I_0^G \to I_1^G \to \cdots .$$

We can fit group cohomology into a bigger picture by observing that $M^G = \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}, M)$ for any $\mathbf{Z}G$-module M, where G acts trivially on $\mathbf{Z}$. The derived functors of Hom are called Ext, and so we have:

$$H^i(G, M) \cong \mathrm{Ext}^i_{\mathbf{Z}G}(\mathbf{Z}, M)$$

[149]. Ext can also be viewed as the left-derived functor of Hom in the first variable, and so group cohomology can be computed using either a projective

resolution of the $\mathbf{Z}G$-module $\mathbf{Z}$ or an injective resolution of M. A useful variant is that if M is a representation of G over a field k, then

$$H^i(G, M) \cong \mathrm{Ext}^i_{kG}(k, M).$$

For example, this definition makes it clear that for a finite group G and prime number p, there is an algorithm to compute any given cohomology group $H^i(G, \mathbf{F}_p)$. It suffices to work out the first $i+1$ steps of a free resolution of $\mathbf{F}_p$ as an $\mathbf{F}_pG$-module,

$$F_{i+1} \to \cdots \to F_0 \to \mathbf{F}_p \to 0,$$

which amounts to doing linear algebra over $\mathbf{F}_p$. Then $H^i(G, \mathbf{F}_p)$ is the cohomology of the chain complex

$$\mathrm{Hom}_{\mathbf{F}_pG}(F_{i-1}, \mathbf{F}_p) \to \mathrm{Hom}_{\mathbf{F}_pG}(F_i, \mathbf{F}_p) \to \mathrm{Hom}_{\mathbf{F}_pG}(F_{i+1}, \mathbf{F}_p).$$

There is a standard free resolution of $\mathbf{Z}$ as a $\mathbf{Z}G$-module that works for any group G [12, vol. 1, section 3.4], but it is usually too big for computations. Rather, the programs that compute the cohomology of finite groups construct a minimal resolution as far as is needed [51, 52].

Example Let $G = \mathbf{Z}/p$ for a prime number p. Write g for a generator of the group G. Let tr be the element $1 + g + \cdots + g^{p-1}$, called the *trace*, in the group ring $\mathbf{F}_pG$. Then $\mathbf{F}_p$ has a free resolution as an $\mathbf{F}_pG$-module that is periodic, of the form

$$\cdots \to \mathbf{F}_pG \xrightarrow[1-g]{} \mathbf{F}_pG \xrightarrow[\mathrm{tr}]{} \mathbf{F}_pG \xrightarrow[1-g]{} \mathbf{F}_pG \to \mathbf{F}_p \to 0.$$

Taking Hom over $\mathbf{F}_pG$ from this resolution to $\mathbf{F}_p$, all the differentials become zero. It follows that $H^i(G, \mathbf{F}_p) \cong \mathbf{F}_p$ for every $i \geq 0$, in agreement with Theorem 1.2.

The low-dimensional cohomology groups have simple interpretations. For any group G and abelian group A, $H^1(G, A)$ can be identified with the abelian group of homomorphisms $G \to A$. Also, $H^2(G, A)$ is the group of isomorphism classes of central extensions of G by A [24, theorem 3.12]. By definition, an *extension* of G by A is a group E with normal subgroup A and a specified isomorphism $E/A \cong G$. It is *central* if all elements of the subgroup A commute with all elements of E.

2

The Chow Ring of a Classifying Space

The Chow groups of an algebraic variety are an analog of homology groups, with generators and relations given in terms of algebraic subvarieties. In this chapter we define Chow groups and state their main formal properties, including a version of homotopy invariance. Using those properties, we define the Chow ring of the classifying space of an algebraic group, a central topic of this book. More generally, we give Edidin and Graham's definition of the equivariant Chow ring of a variety with group action. The chapter ends with a discussion of some open problems about Chow rings of classifying spaces. Examples suggest that the Chow ring of the classifying space of a group is simpler, and closer to representation theory, than the cohomology ring is. But we know much less about general properties of the Chow ring, such as finite generation.

We state the formal properties of Chow groups without proof, using Fulton's book as a reference [43]. Building on that, we develop equivariant Chow groups in more detail. We refer to the papers [138] and [38] for some results, but we do the basic calculations of equivariant Chow groups.

2.1 The Chow group of algebraic cycles

Let us define Chow groups, following Fulton [43]. We work in the category of separated schemes of finite type over a field k. A *variety* over k is a reduced irreducible scheme (which is separated and of finite type over k, by our assumptions). An i-dimensional *algebraic cycle* on a scheme X over k is a finite $\mathbf{Z}$-linear combination of closed subvarieties of dimension i. The subgroup of algebraic cycles *rationally equivalent* to zero is generated by the elements $\sum_D \operatorname{ord}_D(f)D$, for every $(i+1)$-dimensional closed subvariety W of X and every nonzero rational function f on W. The sum runs over all codimension-1 subvarieties D of W, and $\operatorname{ord}_D(f)$ is the order of vanishing of f along D

[43, section 1.2]. The *Chow group* $CH_i(X)$ is the group of i-dimensional cycles modulo rational equivalence.

For a scheme X over the complex numbers, we can give the set $X(\mathbf{C})$ of complex points the classical (Euclidean) topology, instead of the Zariski topology. For X over the complex numbers, there is a natural "cycle map" from the Chow groups of X to the Borel-Moore homology of the associated topological space [43, proposition 19.1.1], $CH_i(X) \to H_{2i}^{BM}(X, \mathbf{Z})$. The Borel-Moore homology of a locally compact space is also known as homology with closed support. The numbering is explained by the fact that a subvariety of complex dimension i has real dimension $2i$. The definition of the cycle map uses the fact that a complex manifold has a natural orientation.

The cycle map is far from being an isomorphism in general. For example, if X is a smooth complex projective curve, then the Chow group of 0-cycles, $CH_0(X)$, maps onto $H_0(X, \mathbf{Z}) = \mathbf{Z}$ with kernel the group of complex points of the Jacobian of the curve. The Jacobian is an abelian variety of dimension equal to the genus of X, and so $CH_0(X)$ is an uncountable abelian group when X has genus at least 1.

A proper morphism $f: X \to Y$ of schemes over a field k determines a pushforward map on Chow groups, $f_*: CH_i(X) \to CH_i(Y)$. A flat morphism $f: X \to Y$ with fibers of dimension r determines a pullback map, $f^*: CH_i(Y) \to CH_{i+r}(X)$. (The morphism f is allowed to have some fibers empty.) Both types of homomorphism occur in the basic exact sequence for Chow groups, as follows [43, proposition 1.8].

Lemma 2.1 *Let X be a separated scheme of finite type over a field k. Let Z be a closed subscheme. Then the proper pushforward and flat pullback maps fit into an exact sequence*

$$CH_i(Z) \to CH_i(X) \to CH_i(X - Z) \to 0.$$

For X a complex scheme, the basic exact sequence for Chow groups maps to the long exact sequence of Borel-Moore homology groups:

$$\cdots \to H_{2i}^{BM}(Z, \mathbf{Z}) \to H_{2i}^{BM}(X, \mathbf{Z}) \to H_{2i}^{BM}(X - Z, \mathbf{Z}) \to H_{2i-1}^{BM}(Z, \mathbf{Z}) \to \cdots.$$

Note the differences between the two sequences. In the exact sequence of Chow groups, we do not say anything about the kernel of $CH_i Z \to CH_i X$. Indeed, the exact sequence of Chow groups can be extended to the left, but that involves a generalization of Chow groups known as motivic homology groups (or, equivalently, higher Chow groups); see Section 6.2. But Chow groups are simpler in one way than ordinary homology: the restriction map to an open subset is always surjective on Chow groups. Geometrically, this is because the closure in X of a subvariety of $X - Z$ is a subvariety of X. This phenomenon lies behind various ways in which Chow groups behave more simply than ordinary homology.

Chow groups are homotopy invariant in the following sense. An *affine bundle* $E \to B$ is a morphism that is locally a product with fibers A^n. We do not assume anything about the structure group of the fibration. So the total space of a vector bundle is an affine bundle, but affine bundles are more general.

Lemma 2.2 *For an affine bundle $E \to B$ with fibers of dimension n, the pullback $CH_i B \to CH_{i+n} E$ is an isomorphism.*

Proof One natural approach uses motivic homology groups, a generalization of Chow groups. Namely, flat pullback gives an isomorphism from the motivic homology of any k-scheme B to the motivic homology of $B \times A^n$ [14, theorem 2.1]. The lemma follows via the localization sequence for motivic homology [15]. (We state the localization sequence for smooth k-schemes as Theorem 6.8.) □

For a smooth scheme X of dimension n over a field k, we write $CH^i(X)$ for the Chow group of codimension-i cycles, $CH^i(X) = CH_{n-i}(X)$. Intersection of cycles makes the Chow groups of a smooth scheme into a commutative ring, $CH^i(X) \times CH^j(X) \to CH^{i+j}(X)$. Fulton and MacPherson's approach to constructing this product first reduces the problem to that of intersecting a cycle on $X \times X$ with the diagonal, and then defines the latter intersection by deformation to the normal cone [43, chapter 6]. Any morphism $f: X \to Y$ of smooth schemes over k determines a pullback map $f^*: CH^*Y \to CH^*X$, which is a homomorphism of graded rings. (When f is flat, this coincides with the flat pullback map $f^*: CH_*Y \to CH_*X$.) For a smooth complex scheme X of dimension n, Poincaré duality is an isomorphism $H^i(X, \mathbf{Z}) \cong H^{BM}_{2n-i}(X, \mathbf{Z})$. So we have a cycle map $CH^*X \to H^*(X, \mathbf{Z})$, and this is a ring homomorphism, sending CH^i into H^{2i}.

Note that homotopy invariance of Chow rings (Lemma 2.2) does not mean that two smooth complex varieties that are homotopy equivalent as topological spaces (in the classical topology) have isomorphic Chow rings. For example, an elliptic curve X over $\mathbf{C}$ is homotopy equivalent, as a topological space, to $Y = (A^1 - 0)^2$. But $CH^1 Y$ is zero by the basic exact sequence of Chow groups (Lemma 2.1), whereas the abelian group $CH^1 X$ is an extension of $\mathbf{Z}$ by the group $X(\mathbf{C}) \cong (S^1)^2$ [67, example II.6.10.2, example IV.1.3.7].

A vector bundle E on a smooth scheme X has Chern classes $c_i E \in CH^i X$, with the same formal properties as in topology. We record the Chow ring of a projective bundle, which is given by the same formula as the cohomology ring of a projective bundle [43, remark 3.2.4, theorem 3.3]:

Lemma 2.3 *Let X be a smooth scheme over a field. Let E be a vector bundle of rank n on X. Let $\pi: P(E) \to X$ be the projective bundle of codimension-1*