INTRODUCTION

Chang and Keisler [8] famously defined model theory as the sum of logic and universal algebra. In the same spirit, one might describe computable model theory to be the investigation of the constraints on information content imposed by algebraic structure. The analogue of the interplay between syntactical objects and the algebraic structure they define is the connection between definability and complexity. One asks: How complicated are the constructions of model theory and algebra? What kind of information can be coded in structures like groups, fields, graphs, and orders? What mathematical distinctions are unearthed when "boldface" notions such as isomorphism are replaced by their "lightface" analogues such as, say, computable isomorphism?

A special case of the following definition was first rigorously made by Fröhlich and Shepherdson [11], following work of Hermann [17] and van der Waerden [40], which itself built on the constructive tradition of 19th century algebra. It was further developed by Rabin [32, 33] and Mal'cev [27].

Definition. Let $\mathcal{L}$ be a computable signature (language), and let $\mathcal{M}$ be an $\mathcal{L}$-structure whose universe is the set of natural numbers. The *degree of* $\mathcal{M}$ is the Turing degree of the atomic (equivalently, quantifier-free) diagram of $\mathcal{M}$.

A structure is *computable* if its degree is $\mathbf{0}$, the Turing degree of computable sets. Equivalently, a structure $\mathcal{M}$ is computable if, uniformly in the symbols of $\mathcal{L}$, the interpretations in $\mathcal{M}$ of the constant symbols, function symbols, and relation symbols of $\mathcal{L}$ are computable. In the Eastern school of computable model theory, the focus has been on *constructivizations*: in Western terminology, a constructivization of a structure $\mathcal{M}$ is an isomorphism between $\mathcal{M}$ and a computable copy of $\mathcal{M}$. A structure $\mathcal{M}$ is said to be *computably presentable*, or *constructivizable*, if it has some constructivization, that is, if it has a computable copy.

Within computable model theory we identify three research programmes.

1. Pure computable model theory considers the effectiveness of model-theoretic constructions. For example, an examination of the standard proof of the compactness theorem reveals that every complete computable (a.k.a. decidable) theory has a computable model, indeed one whose elementary diagram is computable; such structures are called *decidable* (or *strongly constructivizable*). The countable omitting types theorem can be similarly extended [28]. On the other hand, Millar [29] and Kudaibergenov [25] showed that Vaught's "no two models" theorem fails if we consider only decidable models.

Another example of this line of research is the investigation of the effective properties of "special" models. A typical theorem is the characterization of the decidable complete atomic theories that have decidable prime models (Goncharov and Nurtazin [13] and Harrington [16]), depending on the effective properties of the collection of isolated types of the theory. This result has been extended to an extensive analysis of the degrees of prime, saturated, and homogeneous models of decidable theories by several authors (see e.g. [26]). Similarly, an investigation into the computability of countable models of $\aleph_1$-categorical theories is ongoing (see e.g. [1]).

2. Computable structure theory, a more computability-centric approach, looks at the trace left on computability theory, and in particular on the Turing degrees, by their interaction with model theory. Typical here is Knight's result [22] that if $\mathcal{M}$ is not automorphically trivial and a Turing degree $\mathbf{d}$ computes a copy of $\mathcal{M}$, then $\mathbf{d}$ contains a copy of $\mathcal{M}$. In general, one may ask which sets of Turing degrees are *degree spectra*: the collection of degrees of copies of some structure. For example, Slaman [39] and Wehner [41] showed that the collection of nonzero degrees is a degree spectrum.

This approach, pioneered by Ash and Knight, also asks about the relationship between definability of relations on structures and their complexity. The following is a main result [4, 9]: Let R be a relation on a structure $\mathcal{M}$. Then the property that for every isomorphism $f\colon \mathcal{M} \to \mathcal{N}$, the image of R is c.e. in $\mathcal{N}$ is equivalent to the property that R is definable in $\mathcal{M}$ by an effectively presented infinitary Σ^0_1 formula in the logic $\mathcal{L}_{\omega_1,\omega}$. One investigates not only the degrees of structures, but also how complicated are the isomorphisms between structures. This line of research leads to new notions, motivated by computability, which have no analogue in "boldface" model theory. Central among them are the notions of *computable categoricity*, *relative computable categoricity*, and *computable dimension*. (See [2, 3, 15] for definitions and further discussion of these notions.) These are properties of structures rather than theories. A characterization of relative computable categoricity (Ash, Knight, Manasse, and Slaman [4], Chisholm [9]) in terms of definability of the orbits of $\mathcal{M}$ under the action of the automorphism group of $\mathcal{M}$ is an effective version of Scott's analysis of the isomorphism types of countable structures using infinitary logic.

3. Computable algebra investigates the effective properties of particular classes of structures. In some sense this is applied computable model theory. Researchers have attempted, for example, to characterize, among the class of Abelian p-groups, which Abelian p-groups have computable copies. One also asks about the relationship between the complexity of a structure and the complexity of associated objects; for example, Fröhlich and Shepherdson implemented van der Waerden's construction of a computable field with no splitting algorithm. Similarly, one asks about the complexity of the linear independence relation in computable vector spaces; a definitive answer was given by Shore [37]. One asks how complicated are algebraic constructions: for example, the algebraic closure of a computable field has a computable copy, but the image of the original field in its algebraic closure need not always be computable [33]. Instances of notions from computable model theory may have succinct characterizations: Goncharov and Dzgoev [12] and Remmel [34] showed that a computable linear ordering is computably categorical if and only if it contains only finitely many successor pairs. One also asks what are the degree spectra of structures in particular classes, such as linear orderings. For example, Jockusch and Soare [20] showed that there is a low linear ordering with no computable copy; this result was extended by R. Miller [30], though the question of whether the Slaman–Wehner example mentioned above can be realized by a linear order is still an important open problem. On the other hand, every low_4 Boolean algebra has a computable copy [23].

Computable model theory is also related to reverse mathematics, the project of classifying theorems of mathematics in terms of proof-theoretic strength, often by showing equivalence (over a weak base theory) of these theorems with certain subsystems of second order arithmetic (see [38]). For example, the result that a computable field has a computable algebraic closure translates to a proof in the system RCA_0 of recursive comprehension of the existence of algebraic closure of any given field. Thus, computable algebra is often the key for classifying theorems of algebra within reverse mathematics. Similarly, the investigations of pure model theory yield a reverse mathematical classification of theorems of model theory; see for example [18].

While model theory has interesting things to say about countable models (such as Vaught's theorem, or the Ryll-Nardzewski theorem), the real strength of model theory, and in particular stability theory, is apparent in the realm of uncountable models, with Morley's theorem on uncountable categoricity being both a paragon and the catalyst of modern model theory. It is only natural to wish to find the effective content of this part of mathematics. Yet computable model theory has been restricted to investigating countable models, and its interactions with stability theory has been only at the fringe of the latter, for example using the Baldwin-Lachlan analysis [5] of models of uncountably categorical theories to understand effective properties of *countable* models of such theories. Nonetheless, intuitively one sees "effective" and "non-effective" aspects of uncountable model theory and uncountable mathematics in general, and one would like to formalize them and reason about them.

The source of the restriction to countable structures is the fact the the objects that are manipulated by models of computation are hereditarily finite. Turing machines take as input finite strings over a finite alphabet, register machines store natural numbers, and so on. In other words, the world of computability theory is inherently countable. In order to develop an effective theory of uncountable structures, one needs to generalize the theory of computable functions and sets to include uncountable domains. There is no canonical generalization of this sort, and so the kind of effective theory of uncountable mathematics one gets depends heavily on the choice of the model of computation.

The purpose of this book is to describe eight such choices and the resulting applications to the study of effective properties of uncountable objects. It is intended as both an invitation to uncountable computable mathematics and a resource for researchers in the area who, while working along one or more of these lines, are interested in the possibilities raised by other approaches. We first discuss approaches to uncountable computable model theory in particular, via the choice of some model of computation, and then discuss effective uncountable mathematics in greater generality.

In "Borel structures: a brief survey", Montalbán and Nies take "effective" to mean "Borel". They look at structures whose universe is a Borel subset of a Polish space, and where the relations and functions on the structure are uniformly Borel; and also concentrate on Borel homomorphisms between such structures. More generally, they also accept structures that are quotients of Borel structures by a Borel equivalence relation. In this context, Hjorth and Nies [19] verified the failure of an effective compactness theorem, and Nies and Shore [unpublished] have computed the Borel dimension (number of Borel inequivalent Borel copies) of the field of complex numbers to be $2^{\aleph_0}$.

Coskey and Hamkins ("Infinite time turing machines and an application to the hierarchy of equivalence relations on the reals") show what happens if one lets Turing machines run beyond forever; that is, if computations of Turing machines run for an ordinal amount of time. These machines can then be used to compute subsets of Cantor space 2^ω, by writing entire reals on the input tape. The sets of real numbers that can be computed by such machines are all Δ^1_2; all Π^1_1 sets can be so computed. Thus, infinite time Turing machine computation is in a sense an extension of the Borel model. In this context, Hamkins, Miller, Seabold, and Warner [14] showed that the effective version of the completeness theorem is independent of ZFC: it holds if $V = L$, but can be forced to fail, for example in any model in which there are no Σ^1_2 sets of size $\aleph_1$. As in the Borel world, here too there may be a difference between "injective" presentations and presentations that allow for taking a quotient by a computable equivalence relation. Unlike the Borel case, in the context of infinite time Turing machine computability, it is independent of ZFC whether the injective and non-injective notions coincide, that is, whether every structure with a computable presentation has an injective presentation.

Blum, Shub, and Smale [6] introduced a notion of computability over real numbers. In this model, a machine treats a real number as a complete object and does not require an approximation for the number; on the other hand, the machine runs for finitely many steps. Although originally developed with an eye toward modeling numerical analysis, it is natural to consider this notion as a model for computability for sets and functions of real numbers. In "Some results on $\mathbb{R}$-computable structures", Calvert and Porter pursue the development of effective model theory using the Blum–Shub–Smale computation scheme. In particular, they discuss $\mathbb{R}$-vector spaces, 2-manifolds, and homotopy groups.

The concept of Σ-definability was developed by Ershov [10]. We say that a structure $\mathcal{M}$ is *Σ-reducible* to a structure $\mathcal{N}$ if $\mathcal{M}$ is interpretable in the smallest hereditarily finite set containing the elements of $\mathcal{N}$ as ur-elements using existential formulas. In other words, $\mathcal{M} \leqslant_{\Sigma} \mathcal{N}$ if $\mathcal{N}$ can interpret $\mathcal{M}$ effectively when imbued with the power of arithmetic on the natural numbers. In "Effective model theory: an approach via the Σ-definability", Stukachev surveys the Σ-definability approach. A source of unexpected results in this context is the reducibility of fields to linear orderings. For example, Ershov showed that the field of complex numbers is Σ-reducible to any dense linear ordering of size continuum (but is not Σ-reducible to any set, i.e., to any structure with empty signature), whereas the field of real numbers is not Σ-reducible to any linear ordering.

In "Computable structure theory using admissible recursion theory on ω_1", Greenberg and Knight use admissible recursion theory as a model of computation on infinite cardinals, in particular on ω_1. This model has several equivalent definitions, but the original and shortest definition states that computability is given by definability by existential formulas over the structure $(L_{\omega_1}, \in)$, where L is Gödel's constructible universe. This choice allows for a development of computable model theory for structures of size $\aleph_1$ much along the lines of the development of computable model theory for countable structures. The authors discuss, for example, fields, vector spaces, and linear orderings; and pure computable model theory, with a look into the effective completeness theorem, Scott families, and computable categoricity.

The theory of E-recursion is an extension of admissible recursion theory to inadmissible sets. The *divergence-admissibility split* states that the inadmissible sets L_α that are closed under E-recursive functions are exactly those which admit *divergence witnesses*. Thus computability on these inadmissible *E-closed* domains has new properties that are not mirrored in admissible recursion theory. In "E-recursive intuitions", Sacks discusses how the logic $\mathcal{L}_{\alpha,\omega}$ behaves with respect to the completeness and compactness theorems, when L_α is inadmissible and E-closed.

Miller, in "Local computability and uncountable structures", takes a different approach. Rather than using some theory of computation for uncountable sets, local computability measures the effectiveness of uncountable structures by examining their finitely generated substructures and how embeddings between these lift to containments of substructures of the original uncountable structure. This approach yields distinctions between, for example, the field of real numbers and the field of complex numbers; in a sense, the latter is "more" locally computable than the former. Local computability of the real field relies on Artin's theorem, and so requires not insignificant algebra. Miller also discusses linear orderings and trees.

Finally, in "Reverse mathematics, countable and uncountable: a computational approach", Shore discusses uncountable versions of reverse mathematics. While reverse mathematics is rooted in proof theory, computability theorists often prefer to ignore nonstandard models of (first order) arithmetic, and so concentrate on the ω-models (that is, models with standard first-order part) of theorems of mathematics. This approach is generalized by Shore to uncountable domains, using admissible recursion theory as the computational tool. In this context, Shore analyzes a number of statements of uncountable algebra in terms of their computational strength. For example, he shows that the existence of a basis is equivalent to closure under the Turing jump, and that the existence of prime ideals in a ring is equivalent to an uncountable version of weak König's lemma, the finite character tree property.

We would be remiss if we omitted computable analysis from this discussion. It is the longest-standing and most-developed approach to computability on the real numbers, stemming from Turing's own definition of a computable real number and covering a wide range of topics since then. Over that time, many introductions to the subject have been written and are available to the interested reader. Therefore, we did not feel the need to add another one in this volume, but we recommend [7] as a useful and recent tutorial on computable analysis, very much in the style of the textbook [42], and we encourage the reader to keep computable analysis in mind when considering the fifteen questions at the end of this introduction. Among earlier books on the subject, we would also mention [24] and [31]. The approach taken is to view a real number x as given by a Cauchy sequence of rational approximations $\langle q_n \rangle$, converging effectively to x, i.e. with $|x - q_n| < 2^{-n}$ for all n. A real number is computable if there is a computable Cauchy sequence converging effectively to it. One can then define a computable function f on all real numbers to be given by a Turing functional Φ, which uses as oracle a Cauchy sequence (computable or not) converging effectively to the input x, and, on input n, outputs the nth element of a Cauchy sequence converging effectively to $f(x)$. (Such a function is sometimes called *type-two computable*; there are certain analogies to the infinite-time Turing machines in the chapter of Coskey and Hamkins.)

The methods outlined in the eight papers in this book for developing an effective theory of uncountable mathematics are quite distinct. They yield different collections of computable structures and mappings between them. Nevertheless, we would like to discuss some similarities and particularly noteworthy distinctions between them, and some themes to which most of them relate. Some of these issues are special to uncountable mathematics, and do not have analogues in the countable realm.

The most jarring issue, to some, is independence, and the reliance on set-theoretic hypotheses beyond Zermelo–Fränkel set theory. For example, both Shore and Greenberg–Knight, when dealing with objects of size κ, assume that all bounded subsets of κ are constructible, so $H_\kappa = L_\kappa$; this assumption ensures that every subset of κ is amenable for L_κ. Similarly, we mentioned above that basic results in the infinite time Turing machine model are independent of ZFC, essentially because of the fact that this theory goes beyond the Borel world to Δ^1_2 sets. Traditionally, computation is considered a "down to earth" part of mathematics, absolute between models of set theory, and invariant to the choice of axioms of set theory. (Although independence results do occur in classical computability theory, they do not occur in the basic theory, but rather arise in contexts where there is a mix of set theory with computability theory.) Thus, some may expect any theory of computation, including one on uncountable objects, to be basic enough to maintain this invariance.

Another difference between the various approaches is whether they allow considerations of structures of different cardinalities. A number of approaches – Borel computability, infinite time Turing machines, and the Blum–Shub–Smale model – apply only to structures of size the continuum (although the Blum–Shub–Smale model generalizes to work over any ring). On the other hand, Σ-definability and local computability work, at once, for all cardinals. In the middle, admissible recursion theory can work with any cardinal, but requires us to fix a cardinal. That is, admissible κ-recursion theory is defined for each κ, but for distinct cardinals κ and λ, κ-computability and λ-computability are incompatible. The issue of working with distinct cardinals at once may come up, for example, when considering effective versions of the Löwenheim–Skolem theorems, an avenue that is yet unexplored.

In practice, it turns out that another fundamental distinction between models of computation of uncountable objects is the extent to which they have access to a well-ordering of the universe. Traditionally, the ordering and successor relations on the natural numbers are computable. This fact means that searching for witnesses for a particular computable property is a computable procedure. Indeed, the centrality of this aspect of computability to the general theory is evident from Gödel's definition of the class of partial computable functions by the least number operator. This property has profound implications in classical computable model theory. Consider, for example, a (countable) computable field F. Given a polynomial $f \in F[x]$, if we know that f has a root in F, then such a root can be effectively found by searching over the elements of the field and testing them one by one as inputs for f, until a root is found. An ordering of F in order-type ω, which ensures that such a search will end in finitely many steps, cannot be separated from F itself; computable model theory does not know how to "forget" about this ordering of F, and access F only via its algebraic structure. To some mathematicians, this power is unreasonable: algorithms involving an "explicit" field (in the language of van der Waerden) should rely only on algebra, not on external properties such as an ordering of the field. In that view, finding roots of polynomials should be done algebraically, say using Newton's method, but not via "mindless" search.

Borel computation, Blum–Shub–Smale, and infinite time Turing machines have no access to a well-ordering of the continuum, and so according to the view above are "purer" models of computation. Similarly, Σ-definability has access only to the traditional structure on ω; the elements of a structure $\mathcal{M}$ are considered as urelements and are not effectively ordered. Local computability too does not access a well-ordering of uncountable objects, since it directly manipulates only finitely generated substructures of the given structure.

The consequences are dramatic. They are exemplified by the "lost melody" theorem of infinite time Turing machines, which states the existence of a real $c \in 2^\omega$ that can be recognized by an infinite time Turing machine, but not produced (or enumerated) by any such machine that has no access to c as an oracle. In these models of computation, the traditional equivalence between computable enumerability and semi-decidability is broken. In contrast, admissible recursion theory and E-recursion theory work with a computable well-ordering of their universe, although in E-recursion the utility of such an ordering is limited, compared to the unbridled access given to admissible recursion theory. As a result, admissible recursion reflects many of the properties of countable computability, and many constructions of the latter lift to uncountable constructions in the former; but the theory loses the pure reliance on algebra.

The dichotomy between models with a well-ordering of the universe and models without one is a special case of a larger theme: the choice between an intuitive presentation and computational power of a model of computation. Few would dispute that a desirable property of a model of computation is a presentation that makes it intuitive. We would like to understand easily why the model purports to capture the notion of computability. This desideratum explains why it was Turing's machine model, rather than, say, Gödel's equivalent definition of partial recursive functions, that convinced mathematicians that this class of functions correctly captures the notion of computability of functions of natural numbers.

Often, the price of a clear intuitive definition is a weaker theory of effective mathematics. An alternative approach would allow for intuition to develop via usage. Thus, for example, admissible recursion theory has a machine definition via ordinal register machines, or by Turing machines with an ordinal-length tape; but calling these definitions "intuitive" relies, at least, on initially being comfortable with ordinals as basic building blocks. In any case, this is a matter of taste and personal judgment, and so we leave it to the readers.

The above discussion points to some issues that the reader might wish to keep in mind when reading the ensuing papers. Of course, each approach to uncountable computable model theory also has its own specific set of open problems and research directions, discussion of which is best left to the individual papers.

However, there are several other general questions relevant to many or all of these approaches. To finish this introduction, we will list some here, including ones suggested by our previous discussion. In some cases, the answers for particular approaches may be clear or already known, but in many cases, thorough accounts are still challenges for the future of this fascinating area of research.

Here then are some of the questions one might ask when considering an approach to uncountable computable model theory such as the ones discussed in this book.

1. What are the effective versions, under the given approach, of the most basic model-theoretic notions and results, such as the notion of isomorphism and the Completeness Theorem? Moving beyond these, we may consider areas of investigation that have met with great success in the countable case, for example the effective content of the study of special models, such as atomic and homogeneous models, or structure/nonstructure theorems that point out differences between various classes of structures, as in the work of Richter [35, 36].
2. Beyond the above areas, what does the approach have to say about modern concerns in model theory, such as stability theory, which are in many ways beyond the ken of classical computable model theory?
3. Which particular areas of "classical" mathematics is the approach well suited to investigating?
4. There are certain uncountable structures, such as $\mathbb{R}$, which most people would agree are "intuitively effective". Does the given approach make such structures formally effective, or if not, is there a good reason for the mismatch between our intuitions and the formal definitions? The same question may be asked about intuitively effective constructions and theorems.
5. A great part of the success of classical computable mathematics comes from the close connections between computability theory and definability. How well does the given approach interact with definability?
6. How well does the notion of relative computability generalize under this approach, and what impact does the answer to this question have to generalizing notions that such as degree spectra of structures and relations on structures, which in the countable setting rely on the structure of the Turing degrees?
7. In the countable setting, one of the most important tools in analyzing the effective content of mathematics is the theory of classes of degrees, such as low degrees, PA degrees, hyperimmune-free degrees, and so on, which, while usually defined in computability-theoretic terms, have deep connections with combinatorial principles that often arise in the deep analysis of mathematical concepts and constructions. What are the useful analogues to such notions under the given approach, and how may they be applied to the study of uncountable effective mathematics?

8. One reason to be interested in computable mathematics, and the closely related program of reverse mathematics, is that computable procedures (or, in the context of reverse mathematics, the roughly corresponding weak base system RCA_0) may be seen as a mathematically precise version of Hilbert's finitistic procedures (or at least a concept that can play a similar foundational role to finitistic procedures). To what extent does the given approach provide a notion of effective procedure that can reasonably fit the same foundational role?
9. How dependent is the approach on set-theoretic assumptions, and how is it affected by varying these assumptions? Do the differences thus obtained say something of interest about the effective aspects of uncountable mathematics?
10. Is the approach limited to particular cardinalities? If so, can it be generalized to other cardinalities?
11. Is there a way under this approach to consider the effectiveness of structures of different cardinalities at once?
12. To what extent are well-orderings of the universe important to this approach? If such well-orderings of uncountable universes are available, what effects do they have on the results obtained under the approach?
13. Are there ways of extending the approach to address questions of efficiency raised by complexity theory? In this connection, a particularly active area of current research in the countable setting is that of automatic structures (see e.g. [21]).
14. Is there anything the approach can tell us about the countable setting?
15. What are the connections between this approach and other ones? Here we mean both "hard" mathematical connections, allowing us perhaps to transfer results obtained via one approach to another, and comparisons of results. For instance, does effectiveness (of a particular structure or construction, say) under one approach tend to imply effectiveness under the other, at least heuristically? If certain constructions turn out to be effective under one approach but not the other, what does this situation tell us about the nature of these constructions? Turning things around, can such differences in the resulting theories of effective mathematics be used to improve our understanding of the differences between various approaches to "pure" uncountable computability theory?

REFERENCES.

[1] U. Andrews, A new spectrum of recursive models using an amalgamation construction. *J. Symbolic Logic* **76** (2011), 883–896.

[2] C.J. Ash, Isomorphic recursive structures. In *Handbook of Recursive Mathematics*, vol. 1 (Yu.L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel, eds., V. W. Marek, assoc. ed.), Elsevier, 1998, 167–181.