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Finite Ordered Sets

Concepts, Results and Uses

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Contents

Preface

The notions of order, classification, and ranking exist in numerous human activities and situations: administrative or social hierarchies, organization charts, scheduling of sports tournaments, precedence, succession or preference orders, agendas, school, audiovisual or webpage rankings, alphabetical and lexicographic orders, etc. It would be endless to enumerate all the situations where orders appear.

It is thus not surprising, considering the development of the use of mathematics in the modeling of multiple phenomena, to find a great number of fields where order mathematics occur. Nevertheless, the latter are relatively recent. Of course, in mathematics, the notion of the order of magnitude has been known for a long time and in the sixteenth century the symbols "<" and ">" appeared for the first time to express "less than" and "greater than."[1] Yet, the abstract notion of an order defined as a particular type of transitive relation was developed only between 1880 and 1914 by mathematicians and/or logicians such as Peirce, Peano, Schröder, Cantor, Dedekind, Russell, Huntington, Scheffer, and Hausdorff, in the context of the formalization of the "algebra of logic" (that is, Boolean algebra) and also of the creation of set theory (with the study of "order types"). Lattices, which are particular orders since they can be defined algebraically, were also considered as early as the later part of the nineteenth century by Schröder and Dedekind, and then fell into oblivion before arising again during the 1930s thanks to Birkhoff, Öre, and several other eminent mathematicians. For a long time, lattices were the main studied orders. Lattice theory – as well as universal algebra, which is its natural extension – is still extremely active. Besides, the most fundamental result of the theory of finite orders, namely Dilworth's Theorem, was proved only in 1950 in relation to a problem on lattices. However, since the 1970s, the situation has evolved significantly. Researches on order structures have increased widely to answer internal motivations of "pure mathematics" as well as problems raised by the use of these structures in "applied mathematics" (in fields such as operations research, microeconomics, data analysis, data mining, biology, robotics,

[1] In *Artis Analyticae Praxis ad Aequationes Algebraicas Resolvendas*, by the mathematician Thomas Harriot.

theoretical computer science, algorithmics, etc).[2] Today, no less than a treatise of at least 1000 pages would be necessary to present a mere synthesis of the existing results.

This is of course not the purpose of this book, which is limited to some aspects and to the following three main goals:

- to define the concepts and to expound the fundamental results on finite ordered sets;
- to present their uses in various fields;
- to point out a number of results and current works.

The choice to remain within the scope of *finite* ordered sets is in part justified by our concern to publish a reasonably sized book. It also places the book in the field of discrete mathematics, the importance of which nowadays is clear. In this (still very wide) scope, we have given greater importance to the notions and results which seemed essential to us because of their uses in a great number of modelings: linear extensions of an order, isotony, closure operators and closure systems, residual maps and Galois connections, chains and antichains with the Dilworth and Sperner theorems, the duality between ordered sets and distributive lattices, order codings and dimensions, interval orders and semiorders, Arrowian results on orders, etc. And actually, in every chapter of this book, the reader will find examples of uses of order structures in various fields. The last and longest chapter develops some of these uses in (often interdisciplinary) contexts such as preference modeling, data analysis, and scheduling.

At last, in order to cover up the fact that we present only the most fundamental results, each chapter is enriched with a "Further topics and references" section. There, we point out numerous themes that could not be developed in the body of the chapter, and we sometimes give some historical elements, often useful for a better understanding of the subjects.

There is a more general motivation for writing this book. We have mentioned the important development of lattice theory, which has been so much written about – several tens of volumes. On the contrary, books on general ordered sets are very rare and, most often, deal with particular aspects (see Appendix D). A consequence of this situation is that some results are too often unused or rediscovered, which goes against the good use of mathematics.

We continue this preface with a description of the contents of the chapters and appendices.

The principal aim of Chapter 1 is to define and illustrate the fundamental notions used to describe, study, and work on ordered sets, these notions being used and/or developed in the following chapters. Consequently, this chapter presents few results

[2] Another expression of this development was the creation in 1985 of the journal *Order* by Ivan Rival.

and gives no proof. It nevertheless contains examples of uses of ordered sets in fields going from mathematics to operations research, biology, computer science or social sciences.

If orders are binary relations satisfying strong properties, it is however a fact that there exist few results concerning the class of all orders. Actually, like in some other mathematical theories, one is often concerned with classes of orders verifying some particular properties. Chapter 2 describes the most important classes of ordered sets: ranked ordered sets, semimodular and bipartite ordered sets, ordered sets defined by forbidden configurations, semilattices and lattices, linearly ordered sets.

Chapter 3 is devoted to the important question of morphisms between ordered sets; that is, maps between two ordered sets, which preserve or reverse their order. Among these morphisms, one finds codings, closure and dual closure operators, residuated, residual or Galois maps. The latter are the components of Galois connections, the fundamental tool which allows us to set the duality between two ordered sets and, in particular, to define a Galois lattice – the use of which goes from the search for "Guttman scales" (in questionnaires analysis) to the generation of "concepts" (in data analysis or artificial intelligence). This is also the chapter in which we develop the important notions of irreducible elements of an ordered set and of arrow relations between these elements.

Dilworth's Theorem (1950) sets the equality in any ordered set between the maximum number of its pairwise incomparable elements and the minimum number of chains in a chain partition of the ordered set. This is a central result since, on the one hand, it holds for any ordered set and allows us to solve a problem met in various situations (for instance, in operations research, computer science or plane geometry) and, on the other hand, it is related – and in fact, often equivalent – to many other famous results in combinatorics, namely, for example, the König–Hall, Menger, and Ford and Fulkerson theorems. Chapter 4 is devoted to Dilworth's Theorem, and also to the generalizations of another famous result due to Sperner, which gives the maximum number of incomparable subsets (for the inclusion relation) of a given set. Sperner orders, studied in this chapter, are the ranked orders for which the maximum number of incomparable elements of the order can be obtained from the consideration of its rank-sets.

The representation theorem of distributive lattices owing to Birkhoff provides a set representation of a distributive lattice by means of its irreducible elements. This leads to a fundamental duality between distributive lattices and ordered sets, which implies that every result on a distributive lattice can be translated into a result on an ordered set, and conversely. This duality is studied in Chapter 5, where we prove that it is the consequence of a Galois connection between binary relations and families of subsets. We also present another duality between orders and some particular sets of linear orders.

Szpilrajn's 1930 result states that every order can be extended into a linear order (called a linear extension of the order) and allows us to prove that every order is the intersection of all its linear extensions. The dimension of an order is then the minimum number of linear extensions of which it is the intersection. The dimension parameter has been studied intensively for theoretical reasons, but also because it was used in a number of modelings. For instance, it was used as an explanatory model of a preference relation in microeconomics: the partial preference order of an economic agent on a set of commodity bundles is interpreted as resulting from the consideration of several criteria modelized by linear orders. Moreover, trying to determine the dimension of an order is also equivalent to searching for the minimum number of linear orders in the direct product of which it can be coded (that is, in which one can find an isomorphic image of the order). More generally, one can be interested in coding an order in a direct product of chains the length of which is given. If these chains have length 1, the latter operation is equivalent to coding the order by some subsets of a set (in other words, by sequences of 0 and 1), and we then talk about a Boolean coding and the Boolean dimension, notions that were introduced and most studied in computer science. Chapter 6 sets out the fundamental results on these codings and dimensions.

All these chapters are illustrated with examples of uses of ordered sets in various contexts. Our last chapter develops some of these uses. The first two sections of Chapter 7 focus on the notion of a preference, which concerns, among others, cognitive science, microeconomics, operations research, artificial intelligence, and also databases (in which it helps to define effective request languages). We first deal with the modeling of a preference relation, for instance that of an economic agent, when we release the strong hypothesis that the indifference relation should be transitive; the suitable models are then interval orders and semiorders. We next consider the problem of the aggregation of several preference relations into a global preference relation, which should be an order, and we establish a number of "Arrowian" theorems that give prominence to the difficulty of getting a satisfactory result. We carry on with the presentation of ordered models used in mathematical taxonomy: hierarchies, valued hierarchies, median semilattices, partition lattices. The next section focuses on the use of Galois lattices in relational data analysis. We show how, from such a lattice or from its associated closure operator, we can deduce an implicational system allowing us to answer questions like: do subjects having such-and-such characteristics have – or not have – such-and-such other characteristics? As for the fifth section, it presents scheduling problems and some ordinal tools used to deal with them.

Each one of the seven chapters, after its "Further topics and references" section, contains as a last section a list of exercises, which illustrates the notions presented in the chapter and the solutions that have to be sought by anyone who really wants to become familiar with ordered set mathematics. For the majority of these exercises, the solutions will easily be found from the results inside the chapter. For the others, hints and references are provided.

The practical use of the notions and results presented in this book requires being able to answer questions asked on an ordered set modeling some situations, which will in general be done by resorting to a program implementing a resolving algorithm. Appendix A gives some basic notions on the theory of algorithmic complexity, and numerous complexity results for order algorithmics. Appendices B and C are concerned with small orders and the counting of orders. Appendix D provides various documentary indications, while the list of references will allow the reader to get to the results mentioned in the "Further topics and references" section of the chapters. As expected, we also provide an index and a list of symbols.

In each chapter, the definitions, theorems, propositions, corollaries, lemmas, examples, and remarks are numbered $n.p$ where n is the chapter number and p the appearance number in the chapter. For instance, Definition 4.14 is the fourteenth numbered item in Chapter 4.

In each chapter, all figures (respectively, tables) are labeled Figure $n.p$ (respectively, Table $n.p$), where n is the chapter number and p the appearance number in the chapter. For instance, Figure 3.6 is the sixth figure in Chapter 3 and Table 7.3 is the third table in Chapter 7.