1

Concepts and examples

This first chapter covers all basic notions of the theory of finite ordered sets and gives an idea of the various domains in which they are encountered. Beware! It would be fastidious and unproductive to approach this book with a linear reading of this chapter. The reader is invited to use it as a reference text in which he will find the definitions and illustrations of the notions used in the next chapters. In particular, we do not provide in this chapter the proofs of the few stated results (the reader will find these proofs in other chapters and/or in exercises). In Section 1.1 we give the concepts and the vocabulary allowing us to define, represent, and describe an ordered set. We also introduce several graphs (comparability, incomparability, covering, neighborhood graphs) associated with an ordered set. Section 1.2 presents some examples of ordered sets that appear in various disciplinary fields from mathematics themselves to social sciences and ranging from biology to computer science. We define the notions of an ordered subset, a chain, an antichain, and of an extension of an ordered set in Section 1.3 and the notions of a join and a meet, of irreducible elements, and of downsets or upsets in Section 1.4. Finally, Section 1.5 describes the basic construction rules (linear sum, disjoint union, substitution, direct product, etc.) that form new ordered sets from given ones.

1.1 Ordered sets

In the very beginning there was the order... or the strict order! This section therefore begins with the definition of these two order notions, with their associated terminology. Then we present different graphs (comparability, incomparability, covering, neighborhood graphs) associated with an ordered set. Section 1.1.1 is devoted to a very useful representation of an ordered set called its diagram. Finally, we end the section by the standard mathematical notion of an isomorphism between ordered structures together with the also very significant notion of a dual isomorphism (or duality).

1.1.1 Orders and strict orders

What is an order in mathematics? This question, raised in 1903 by the logician and philosopher Bertrand Russell, has essentially received two answers (generally) named *order* and *strict order* (see Section 1.6 for the history of these notions). On the other hand, the notations and terms used to refer to the same ordinal notions have been – and remain – very diverse. In this book we do not refrain from using several different symbols or terms to denote or to name the same notions since, from experience, we know that using a unique notation system may cause more disadvantages than advantages. However in this section and in order to compensate for these possible disadvantages, we specify (in a very thorough and thus somewhat tedious way) the two fundamental notions of order with the various notation systems that we will use.

A *binary relation* on a set X is a subset R of the set X^2 of the ordered pairs of X. The notation $(x,y) \in R$ (or xRy) means that the ordered pair (x,y) belongs to the relation R. We write $(x,y) \notin R$ – or xR^cy – if not.

Definition 1.1 A binary relation O on a set X is an *order* if it satisfies the following three properties:

1. Reflexivity: for each $x \in X, xOx$.
2. Antisymmetry: for all $x,y \in X$, xOy and yOx imply $x = y$.
3. Transitivity: for all $x,y,z \in X$, xOy and yOz imply xOz.

The order O is *linear* (or *total*[1]) if, for all $x,y \in X$, xO^cy implies yOx.

An *ordered set* (or a *partially ordered set* or a *poset*) is an ordered pair $P = (X,O)$ where X is a set and O an order on X (sometimes to avoid ambiguity, we will find it useful to denote O_P the order of the ordered set P). If O is a linear order, $P = (X,O)$ is then called a *linearly ordered set* (or a *totally ordered set* or a *chain*). The symbols $\underline{n}$ or C_n denote a chain of size n.

Example 1.2 Let $X = \{a,b,c,d,e\}$ and $P = (X,O)$ be the ordered set where O is the following order on X:

$$O = \{(a,b),(a,e),(c,b),(c,d),(c,e),(d,e),(a,a),(b,b),(c,c),(d,d),(e,e)\}$$

An ordered set P can be represented by a network, the points of which correspond to the elements of X and the arcs (or directed edges) to the ordered pairs of O, the loops representing the ordered pairs of the form (x,x) (cf. Figure 1.1). We can also represent it by tables (see Table 1.1). The cells of these tables correspond to all ordered pairs of X and a 1 or a $\times$ in a cell (respectively, a 0 or an empty cell) means that the corresponding ordered pair belongs (respectively, does not belong) to O. Yet we will

[1] A binary relation R on X is said to be *total* (respectively, *weakly total*) if, for all $x,y \in X$, xR^cy implies yRx (respectively, $x \neq y$ and xR^cy imply yRx).

Table 1.1 Two kinds of table representing the ordered set P in Example 1.2

	a	b	c	d	e
a	×	×			×
b		×			
c		×	×	×	×
d				×	×
e					×

	a	b	c	d	e
a	1	1	0	0	1
b	0	1	0	0	0
c	0	1	1	1	1
d	0	0	0	1	1
e	0	0	0	0	1

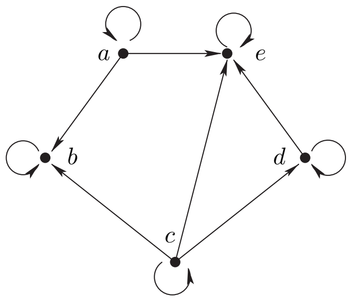


Figure 1.1 An ordered set P represented by a network.

see in Section 1.1.3 a much more economical way to represent an ordered set: the *(Hasse) diagram*.

We now give the definition of a strict order.

Definition 1.3 Let O be a binary relation on a set X.

- O is a *strict order* if it is irreflexive (for each $x \in X$, xO^cx) and transitive. A *strictly ordered* set is an ordered pair $P = (X, O)$, where X is a set and O a strict order on X.
- A strict order O is *strictly linear* if, for all $x, y \in X$, $x \neq y$ and xO^cy imply yOx. We then say that $P = (X, O)$ is a *strictly linearly ordered set*. If $|X| = n$, P or the corresponding strictly linear order may be denoted by $\underline{n}_s$.

Note A strict order O on X is an *asymmetric* relation, i.e., such that yO^cx for all $x, y \in X$ satisfying xOy (prove it).

Since there exists an obvious one-to-one correspondence between the set of orders and the set of strict orders defined on a set X (what is it?), there are two equivalent ways to formalize the notion of an order (see the further topics in Section 1.6). So, to each particular class of orders corresponds a particular class of strict orders (for example, strictly linear orders correspond to linear orders). In order to simplify the terminology, it will sometimes be preferable to use the same terms to name the orders of these two corresponding classes. Thus, in Section 7.1 of Chapter 7, where we will consider several models of strict preferences represented by strict orders, the qualifier "strict" will be systematically omitted.

So far we have used the notation O for an order but, in most cases, this notation is favorably replaced by the symbol "$\leq$" which is read "less than or equal to." We thus use the traditional symbol of the order between numbers for an arbitrary order. An ordered set is then denoted by $P = (X, \leq_P)$ or, more simply, by $P = (X, \leq)$.

Likewise, a strict order will often be denoted by the symbol "$<$," which is read "less than" (or "smaller than"), and we will write $P = (X, <_P)$ or, more simply, $P = (X, <)$ for a strictly ordered set.

Let $P = (X, \leq)$ (or (X, O)) be an ordered set.

- The size of P is the size of X and we may denote it by $|P|$, $|X|$ or simply n according to the context.
- The expression "x belongs to P" means $x \in X$ and we also write $x \in P$.
- The expression "(x,y) belongs to P" means $x \leq y$ (or $(x,y) \in O$) and we also write $x \leq_P y$, xO_Py or simply xOy depending on the notations used for P.
- The number of the ordered pairs belonging to O is denoted by $|O|$, $m(P)$ or simply m.

Let x, y be two elements of an ordered set $P = (X, \leq)$.

- If $x \leq y$, we say that x is *less than or equal to* y, or that y is *greater than or equal to* x. We also say that x is a *lower bound* of y and that y is an *upper bound* of x. The set $\{t \in P : t \leq x\}$ of lower bounds of x is denoted by $(x]$ or Px. The set $\{t \in P : x \leq t\}$ of upper bounds of x is denoted by $[x)$ or xP.
- If $x \leq y$ does not hold, we say that x *is not less than or equal to* y and we write $x \not\leq y$. This relation is also denoted by $\leq^c$ (since it is the complementary relation of the relation $\leq$).
- If $x \leq y$ and $x \neq y$, we say that x is *less than* y, or that y is *greater than* x, and we write $x < y$. We also say that x is a *strict lower bound* of y and that y is a *strict upper bound* of x. The relation $<$ is the strict order relation associated with the relation $\leq$. The set of strict lower bounds (respectively, strict upper bounds) of x is denoted by $(x[$ (respectively, $]x)$).
- If $x \leq y$ or $y \leq x$, we say that x and y are *comparable*. If not, i.e., if $x \not\leq y$ and $y \not\leq x$, we say that x and y are *incomparable* and we write $x||y$ (or $xInc_Oy$ if the order is denoted by O).

In Example 1.2, a and b are comparable whereas a and c are incomparable.

One will observe that $y \not\leq x$ is equivalent to ($x < y$ or $x||y$). It results from the above definitions that a chain is an ordered set in which any two elements are always comparable. Conversely, we define the notion of an antichain.

Definition 1.4 An *antichain* is an ordered set such that any two distinct elements are always incomparable. We write A_n for an antichain of size n.

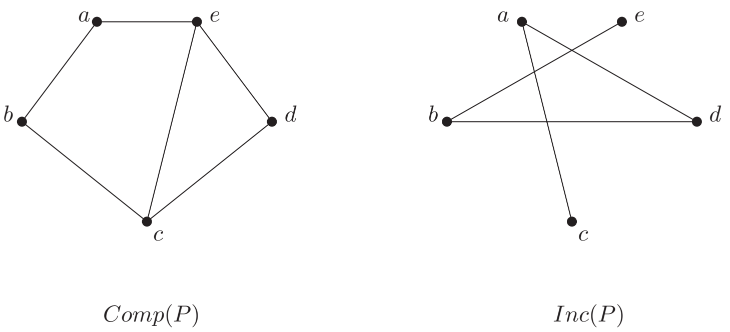


Figure 1.2 The comparability and incomparability graphs of the ordered set in Example 1.2.

1.1.2 Graphs associated with an ordered set

Several graphs are naturally associated with an ordered set,[2] such as, in particular, the comparability, incomparability, covering, or neighborhood graph. Each of these graphs corresponds to some particular aspects of the ordered set and may be important for its study. We define and illustrate these graphs below.

Definition 1.5 Let $P = (X, O)$ be an ordered set. The *comparability graph* of P is the undirected graph $Comp(P) = (X, Comp_P)$, the vertices of which are the elements of P and where the edges are the pairs $\{x,y\}$ of comparable elements in P. The relation $Comp_P$, also written $Comp_O$, is called the *comparability relation of* P.

The *incomparability graph* $Inc(P) = (X, Inc_P)$ of P is defined similarly, with its edges equal to the pairs $\{x,y\}$ of incomparable elements in P. The relation Inc_P, also written Inc_O, is called the *incomparability relation of* P.

Figure 1.2 shows the comparability and incomparability graphs of the ordered set in Example 1.2.

The graphs $Comp(P)$ and $Inc(P)$ are obviously complementary to each other in the sense that the pair $\{x,y\}$ is an edge of one of them if and only if it is not an edge of the other one. Then, to study one of them is equivalent to studying the other.

We now define the covering relation associated with an ordered set. This relation is generally not an order but is, on the other hand, the most economical way (with respect to the number of ordered pairs) to describe an order. It will be used constantly throughout the book.

[2] An *undirected graph* is an ordered pair $G = (X, E)$ where X is a set and E a set of pairs of distinct elements of X, called the *edges* of G; a *directed graph* is an ordered pair $G = (X, A)$ where X is a set and A a set of ordered pairs of X, called the *arcs* of G.

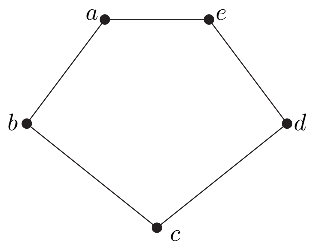


Figure 1.3 The neighborhood graph of the ordered set P in Example 1.2.

Definition 1.6 The *covering relation* of an ordered set $P = (X, \leq)$, denoted by $\prec_P$ or simply $\prec$, is defined by $x \prec y$ if $x < y$ and $x \leq z < y$ imply $x = z$. We then say that x *is covered by* y or that y *covers* x. We write $xP^+ = \{t \in P : x \prec t\}$ and $P^- x = \{t \in P : t \prec x\}$.

The directed graph $Cov(P) = (X, \prec)$ associated with the covering relation is called the *covering graph* of P.

In other words, x is covered by y in P if $x < y$ and if there does not exist in P any element z greater than x and less than y.

The ordered set P in Example 1.2 has five covering ordered pairs: $a \prec b, a \prec e$, $c \prec b, c \prec d$, and $d \prec e$.

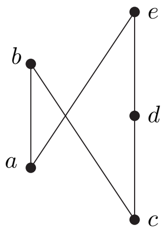
The covering relation of a chain defined on a set of size n is written $x_1 \prec x_2 \prec \ldots \prec x_n$, which we more simply denote by $x_1 x_2 \ldots x_n$. This particularly economical notation of linear orders will often be used. Conversely, any sequence of n distinct elements (or, equivalently, any permutation on these elements) can be seen as defining a linear order on these elements, namely the apparition order in the series. This implies that the number of linear orders on a set of size n is equal to $n!$.

With the covering relation of P is also associated an undirected graph called the *neighborhood graph of* P, denoted by $Neigh(P) = (X, N_P)$, where the pair $\{x, y\} \in N_P$ if ($x \prec y$ or $y \prec x$). For the ordered set in Example 1.2, this graph is given by Figure 1.3.

A number of notions on ordered sets may be defined by means of the neighborhood graph. It is the case for the notion of the connectivity:

Definition 1.7 An ordered set P is *connected* if its neighborhood graph is connected, i.e., if, for any pair of distinct vertices $\{x, y\}$ of P, there exists a sequence $x = x_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_p = y$ of vertices such that $x_i N x_{i+1}$, for any $i = 0, \ldots, p-1$.

A non-connected ordered set is partitioned into maximal connected ordered subsets (see Definition 1.26), called its *connected components*. Since a problem on a non-connected ordered set most often comes back to a problem on its connected components, it is generally enough to consider connected ordered sets.



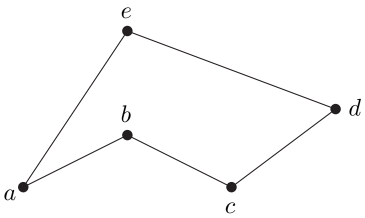


Figure 1.4 Two diagrams of the ordered set in Example 1.2.

1.1.3 Diagram of an ordered set

At the beginning of the chapter we saw that one can consider an ordered set as a network, which is a "geometrical" representation of the latter (see Figure 1.1). However, as soon as one considers a "big" ordered set, the network risks becoming quite inextricable. It is possible to do better thanks to the notion of a diagram of an ordered set, which provides a much more economical representation of an ordered set. First, we observe that to know the covering ordered pairs of $P = (X, \leq)$ allows us to find all ordered pairs of the order $\leq$. Indeed, we have $x < y$ if and only if there exists a sequence $x_0, x_1, \ldots, x_p$ of elements of X such that $x = x_0 \prec x_1 \prec \ldots \prec x_p = y$. Thus we can represent an ordered set thanks to its covering ordered pairs, which is done by using the (Hasse) diagram.

Definition 1.8 The *diagram* (or *Hasse diagram*) of an ordered set $P = (X, \leq)$ is a representation of its covering graph in which the elements x of P are represented by points $p(x)$ of the plane, with the following two rules:

- If $x < y$ (the horizontal line going through) $p(x)$ is below (the horizontal line going through) $p(y)$.
- $p(x)$ and $p(y)$ are linked by a line segment if and only if $x \prec y$.

Clearly, there exist an infinity of possible diagrams for a given ordered set. Yet, just like we did in the above definition, we will generally talk about "the" diagram of an ordered set P, instead of specifying "one of the diagrams" of P. The choice of the position of the points allows us to obtain some diagrams that are easier to read than others. Figure 1.4 shows two possible diagrams for the ordered set in Example 1.2, and Figure 1.5 shows diagrams of the chain C_4, of the antichain A_4, and of the "cube" B_3 (the letter "B" stands for "Boolean," see Example 1.12 further on).

Later on, all figures representing an ordered set will show a diagram of the latter. Let us observe that, since the diagram of an ordered set does not represent the reflexive ordered pairs, it may as well represent the associated strict ordered set.

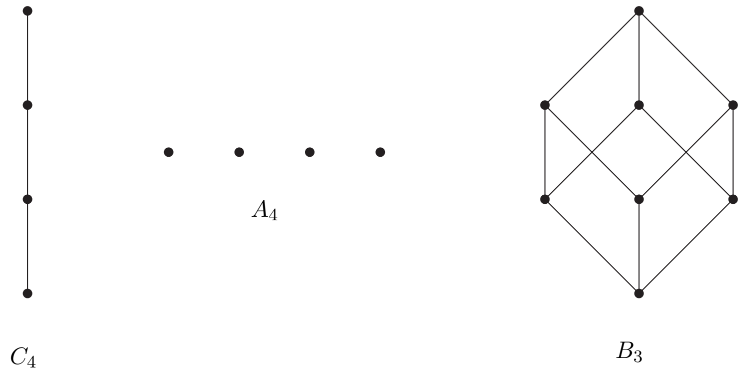


Figure 1.5 The diagrams of the chain C_4, of the antichain A_4, and of the cube B_3.

1.1.4 Isomorphism and duality

In mathematics, the notion of an *isomorphism* between two structures is fundamental. It allows us to prove that two sets of objects of different nature may satisfy the same properties. When considering order structures, we have to consider also the other very significant notion of a *dual isomorphism* (or of a *duality*). Recall that a map f from X to Y is called a *bijection* – or a *one-to-one correspondence* – if it is injective – or one-to-one, i.e., $x \neq y$ implies $f(x) \neq f(y)$, and surjective – or onto, i.e., $f(X) = Y$.

Definition 1.9 Two ordered sets $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ are said to be *isomorphic* (or *of the same type*) if there exists a bijection f from X to Y such that:

$$x \leq_P y \iff f(x) \leq_Q f(y)$$

The bijection f is called an *order isomorphism* between P and Q and we write $P \equiv Q$. When $P = Q$, we say that f is an *automorphism* of P.

In other terms, two ordered sets are isomorphic if they are identical up to the denomination of their elements. Thus we obtain an ordered set isomorphic to that in Figure 1.4 (Example 1.2) by replacing a, b, c, d, e with 1, 2, 3, 4, 5.

The isomorphism relation between ordered sets is an equivalence relation the classes of which, in accordance with Definition 1.9, are called the types of ordered sets. Then two isomorphic ordered sets are said to be "of the same type." In order to illustrate the difference between order and order type, we can note that there exist 130 023 distinct orders defined on a set of size 6 whereas there are only 318 different order types on such a set[3] (for easy countings, see Exercise 1.1). Appendix B provides the diagrams of the 58 connected order types of size at most equal to 5.

[3] In general, counting all orders (respectively, all order types) on a set of size n is very difficult and, at the present time, the answer is known only for $n \leq 18$ (respectively, $n \leq 16$). These numbers increase very quickly (see Appendix C).

Cambridge University Press
978-1-107-01369-8 - Encyclopedia of Mathematics and its Applications: Finite Ordered Sets: Concepts, Results and Uses
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Excerpt
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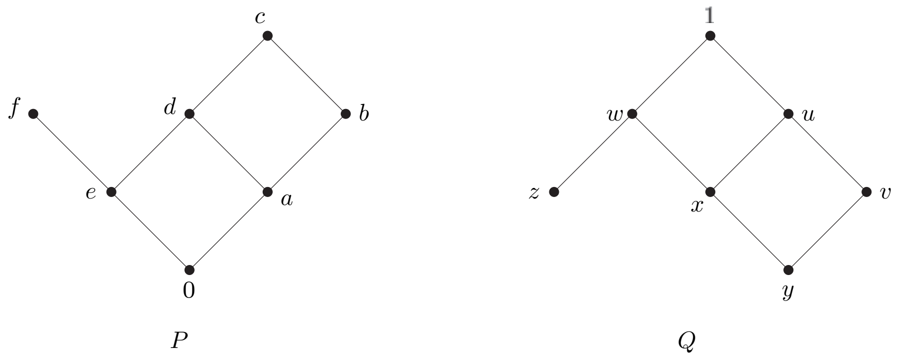

Figure 1.6 Two dual ordered sets P and Q.

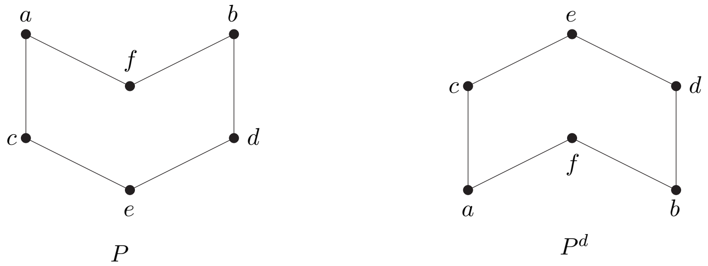

Figure 1.7 An ordered set P and its dual ordered set P^d.

Definition 1.10 Two ordered sets $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ are said to be *dually isomorphic* (or simply *dual*) if there exists a bijection f from X to Y such that, for all $x, y \in X$:

$$x \leq_P y \iff f(x) \geq_Q f(y)$$

The bijection f is called an *(order) dual isomorphism* (or a *dual isomorphism*) between P and Q and we write $P \equiv_d Q$.

See Figure 1.6 for an example of dual ordered sets.

A particularly interesting case of a dual isomorphism is obtained by considering the ordered set $P^d = (X, \leq^d)$, *dual* of an ordered set $P = (X, \leq)$ and defined by:

$$x \leq^d y \iff y \leq x$$

The reader will check that $\leq^d$ is an order and that P and P^d are dually isomorphic. The order $\leq^d$ is called the *dual* (sometimes the *reverse*) of the order $\leq$ and we also denote it by $\geq$. A diagram of P^d is obtained by turning a diagram of P "upside down" (Figure 1.7).

We note that, in the case of a linearly ordered set L which is written $L = x_1x_2...x_n$, the linearly ordered set L^d is written $L^d = x_n...x_2x_1$.

From the existence of the dual order for any order follows the so-called *duality principle* for ordered sets, which states as follows:

If a property using symbols $\leq$ *and* $\geq$ *holds in any ordered set, so does the dual property obtained by permuting these symbols.*

For instance, in any ordered set, any element is less than or equal to at least one maximal element (see page 23). Dually, in any ordered set, any element is greater than or equal to at least one minimal element (see the same page).

More generally, the *dual class* $\mathcal{E}^d$ of a class $\mathcal{E}$ of ordered sets is formed from all ordered sets P^d with $P \in \mathcal{E}$. If a property holds in any ordered set of $\mathcal{E}$, the dual property holds in any ordered set of $\mathcal{E}^d$.

A class $\mathcal{E}$ of ordered sets is said to be *ipsodual* (or *autodual*) if any ordered set of $\mathcal{E}$ has its dual in $\mathcal{E}$, i.e., if $\mathcal{E} = \mathcal{E}^d$ (the class of linearly ordered sets on the one hand, and that of all ordered sets on the other hand, are two examples of ipsodual classes). For such a class, the duality principle then states as follows:

If a property holds in any ordered set of an ipsodual class of ordered sets, so does the dual property.

1.2 Examples of uses

Classifying, comparing, and hierarchizing activities are consubstantial to cognitive activity, so it is not surprising that mathematical models of order are present in a great number of fields, ranging from mathematics to biology, computer science or social sciences. In this section we present a sample of examples where orders – or order notions – appear in the latter fields. Chapter 7 will develop several of these uses. Of course, one could find many other examples: e.g. the uses of orders in quantum theory (see, for instance, Marlow, 1978) and in environmental sciences and chemistry (Brüggemann and Carlson, 2006).

1.2.1 Mathematics

Example 1.11 The notation $\leq$ used for an arbitrary order is the classic notation of the order defined on a set of numbers, for instance on the set $\mathbb{N}$ of non-negative integers.

Another order defined on $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ playing a significant role in number theory is the *divisibility order* on positive integers, denoted by $|$ and defined by: $a|b$ if a divides b (the reader can check that this relation is indeed an order). Any set of positive integers is an ordered set for this divisibility order (see Exercise 1.10 in Section 1.7).

Example 1.12 We denote by $P(E)$ the set of all subsets of a set E. In this book, the notation $\underline{2}^E$ will often stand for the set $(P(E), \subseteq)$ of all subsets of E ordered by