1 Introduction

1.1 Contents of book

An introduction to the use of anisotropic linear elasticity in determining the static elastic properties of defects in crystals is presented. The defects possess different dimensionalities and span the defect spectrum. They include:

- Point defects (vacancies, self-interstitials, solute atoms, and small clusters of these species),
- Line defects (dislocations),
- Planar defects (homophase and heterophase interfaces),
- Volume defects (inhomogeneities and inclusions).

To avoid confusion, an inclusion is defined as a misfitting region embedded within a larger constraining matrix body, and, therefore, acts as a source of stress. It may be either homogeneous (if it possesses the same elastic properties as the matrix) or inhomogeneous (if its elastic properties differ). On the other hand, an inhomogeneity is simply an embedded region with different elastic constants but no misfit.

Following the preliminaries of the present chapter, the book presents (Chapter 2) a concise account of the basic elements of anisotropic and isotropic linear elasticity, and, in addition, derivations of a number of special relationships needed throughout the text. This is followed by a review of methods of solving defect elasticity problems (Chapter 3), derivations of useful Green's functions (Chapter 4) and the basic formulation of interactions between defects and imposed stress in the form of interaction energies and forces (Chapter 5). In Chapters 6 to 15, attention is focused on the individual defects in the following order: inclusions, inhomogeneities, point defects, dislocations, and interfaces. In most cases, the elastic field associated with the defect in an infinite homogeneous region is treated first. Then, the interaction of the defect with imposed stress is studied. This sets the stage for analyzing the behavior of the defect in finite homogeneous regions where interfaces and associated image stresses are present. Finally, in the concluding chapter (Chapter 16), a selection of interactions between various pairs of defects is analyzed.

1.2 Sources

Important sources for the book include the pioneering work of J. D. Eshelby, especially Eshelby (1951; 1954; 1956; 1957; 1961), who invented imaginary cutting, straining and bonding operations to create defects in a manner that greatly expedites the analysis of their elastic properties. By applying potential theory to the results of these operations and using harmonic and biharmonic potentials and the divergence theorem (Gauss's theorem),[1] expressions for defect interaction energies and forces on defects are obtained in the form of integrals over surfaces enclosing the defects. The approach has connections with classic electrostatics and electromagnetism and produces an arsenal of general expressions that can be employed to treat specific defect problems. Other sources include the indispensable treatise of Bacon, Barnett, and Scattergood (1979b), which demonstrated that the anisotropic elasticity theory can often be applied to defects with almost the same ease as isotropic theory, and the more recent book, *Elastic Strain Fields and Dislocation Mobility*, edited by V. L. Indenbom and J. Lothe (1992). Additional valuable sources include the books of Leibfried and Breuer (1978) on point defects, of Teodosiu (1982) on point defects and dislocations, of Hirth and Lothe (1982) on dislocations and of Mura (1987) on inclusions, dislocations, and cracks. The book of Sutton and Balluffi (2006) provided a source for material on interfaces. Finally, many journal articles must be cited, especially those of J. Lothe and D. M. Barnett, dealing with the anisotropic theory.

1.3 Symbols and conventions

The Roman and Greek symbols that are used most frequently are identified in lists before the main text. Components of vectors and tensors are generally indicated by subscripts, while the entities to which various quantities refer to are usually indicated by superscripts: the superscripts of most importance are also listed.

Cartesian coordinates and index notation involving either Latin or Greek subscripts are mainly employed. For Latin subscripts, which run from 1 to 3 (unless noted otherwise), the standard *repeated index summation convention* is employed. Here, any indexed quantity possessing a repeated subscript is automatically summed with respect to that subscript as it runs from 1 to 3, e.g.,

$$x_{ii} = x_{11} + x_{22} + x_{33}, \quad x_i x_i = x_1 x_1 + x_2 x_2 + x_3 x_3, \quad y_{jk} z_k = y_{j1} z_1 + y_{j2} z_2 + y_{j3} z_3. \tag{1.1}$$

[1] Eshelby has been quoted (Bilby, 1990) as saying about this work, "Amusing applications of the theorem of Gauss."

For Greek subscripts, this convention does not apply. Instead, summation of quantities with Greek subscripts is indicated explicitly by the usual summation symbol, e.g.,

$$\sum_{\alpha=1}^{3} x_{\alpha\alpha} = x_{11} + x_{22} + x_{33} \neq x_{\alpha\alpha}. \tag{1.2}$$

Cylindrical (r, θ, z) and spherical (r, θ, ϕ) orthogonal curvilinear coordinates are also employed, and basic formulae referred to these coordinates, rather than Cartesian coordinates, are presented in Appendices A and G.

Complete descriptions of the elastic fields derived throughout the book, i.e., the displacements, strains, and stresses, are normally not all presented together. Instead, to save space, results are presented in forms that can be used to obtain the complete descriptions relatively easily by employing standard relationships between the various quantities. For example, when only the displacement field is given, the corresponding strain field can be determined by simple differentiation, and then the stress field can be obtained using Hooke's law.

Unless noted otherwise, it can be assumed that the results presented throughout the book are valid for general anisotropic systems. Results that are valid only for isotropic elasticity are clearly identified to avoid any confusion.

1.4 On the applicability of linear elasticity

Linear elasticity is an approximation that describes a homogeneous crystal as a uniform continuum in which the stress is proportional to the strain via constant elastic coefficients. For many defect applications, this approximation is quite adequate. It is most reliable in regions that are large enough to span a significant number of atoms and where the atom displacements are small and consequently proportional to the forces exerted on them. With this assumption, the effects of the displacements associated with the solution for one elastic displacement field on the solution for a superposed second displacement field can be neglected. The stresses and strains obtained as a solution of one boundary value problem can then be simply added to the solution of another problem involving other boundary conditions; i.e., linear superposition holds for both the boundary conditions and the solutions.

However, many of the defects of interest, such as point defects and dislocations, possess *core regions* of atomic dimensions where the atoms have undergone relatively large displacements out of the linear elastic range and find themselves in alien atomic environments. As discussed by Read (1953), such highly disturbed material, in which atoms are not surrounded by their usual neighbors, and for which the linear continuum model breaks down, may be regarded as *bad material*, in contrast to *good material*, which corresponds to defect-free crystalline material that is, at most, elastically strained. The core region of a vacancy, for example,

consists of a small roughly spherical region of bad material centered on the vacant site, where neighboring atoms have relaxed and undergone relatively large displacements. The core of a dislocation line consists of a long narrow cylindrical region of bad material, and the core of an incoherent grain boundary consists of a thin plate-like region of bad material in the transition region between the two adjoining bulk crystals.

In view of this, the displacement field of the defect can be broken down into the relatively small core region, where the linear theory cannot be applied, and the much larger surrounding matrix region, where it serves as a good approximation. A quite reliable solution can then be obtained by employing a hybrid approach, in which the displacements in the core are determined by means of atomistic calculations and are matched to the displacements in the adjoining bulk matrix region determined by using linear elasticity. Fortunately, such a complex calculation can be avoided in many situations by realizing that the displacements due to the defect generally decrease rapidly with distance into the matrix, and, at distances several times the relevant core dimension, become insensitive to the detailed nature of the conditions at the core–matrix interface. An acceptable solution for the elastic field in the matrix region beyond a few core dimensions can then be obtained by the exclusive use of linear elasticity with the core described, at most, by a few simple parameters. Since the relevant core dimensions are relatively small, the regions that can be treated in this manner in bodies containing defects typically extend over essentially the entire body and have dimensions corresponding to length scales that are of major interest. This limitation is therefore not a major drawback under many circumstances. The difficulties of dealing with the large non-linear displacements at defect cores can be mitigated to a degree by employing *non-linear elasticity*, but this will not be considered in this book.

A further complication with the use of linear elasticity occurs when abrupt step-like changes in bulk elastic constants are present in a system, as, for example, at the interface between an inhomogeneous inclusion and the matrix. The assumption that the bulk elastic constants in the matrix and inclusion are truly constant right up to the interface is an approximation, since at small distances from the interface the elastic constants of the inclusion and matrix must be affected to at least some degree by their altered local environments, even under conditions when the atomic displacements are relatively small. This problem can be dealt with by employing *size-dependent elasticity*, where it is assumed that the elastic constants depend upon the local environment over a specified length scale (Eringen, 2002; Sharma and Ganti, 2003). This approach introduces additional complexities, however, and will not be considered in this book.

2 Basic elements of linear elasticity

2.1 Introduction

The basic elements of anisotropic linear elasticity are presented in concise form. First, the deformation of an elastically strained body is described in terms of the local displacements, strains, and rotations that occur throughout the body. Requirements on the strains that ensure compatibility of the medium are then described. Next, the forces acting throughout the body are described in terms of surface tractions, body forces, and stresses. Conditions for mechanical equilibrium are derived. The stresses and strains are then linearly coupled via elastic constants, and various stress–strain relationships are derived. Finally, the energy stored in an elastically strained medium is formulated. Elements of the theory for the special case when the medium is elastically isotropic are included,[1] along with several formulations of additional elastic quantities required for treating crystal defects.

References include: Love (1944); Sokolnikoff (1946); Muskhelishvili (1953); Nye (1957); Lekhnitskii (1963); Bacon, Barnett and Scattergood (1979b); Soutas-Little (1999); Hetnarski and Ignaczak (2004) and Asaro and Lubarda (2006).

2.2 Elastic displacement and strain

2.2.1 Straining versus rigid body rotation

When a body is elastically deformed, points within the body are generally displaced by differing degrees: local regions must therefore be strained (deformed) in various ways. To analyze the connection between the displacements and the strains, a Cartesian coordinate system is adopted with unit base vectors $\hat{\mathbf{e}}_i$ and coordinates x_i and its origin fixed in the material. As illustrated in Fig. 2.1, a point

[1] The theory of elasticity presented in this chapter holds for systems in which all displacements and strains are purely elastic. In Chapter 3, stress-free *transformation strains* are introduced as a means of mimicking crystal defects. For systems containing transformation strains the present purely elastic formulation must therefore be modified as described in Section 3.6.

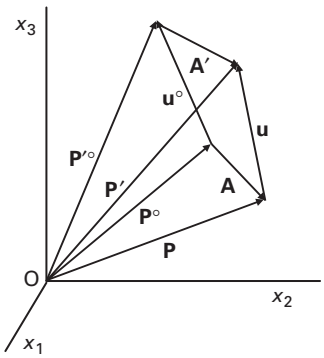

Figure 2.1 Displacements $\mathbf{u}^\circ$ and $\mathbf{u}$ of points initially located at the vector positions $\mathbf{P}^\circ$ and $\mathbf{P}$, respectively.

initially at the position $\mathbf{P}^\circ = x_i^\circ \hat{\mathbf{e}}_i$ is then displaced by the vector $\mathbf{u}^\circ = u_i^\circ \hat{\mathbf{e}}_i$ to the position $\mathbf{P}'^\circ = x_i'^\circ \hat{\mathbf{e}}_i$, while a closely adjacent point initially at $\mathbf{P} = x_i \hat{\mathbf{e}}_i$ is displaced by $\mathbf{u} = u_i \hat{\mathbf{e}}_i$ to $\mathbf{P}' = x_i' \hat{\mathbf{e}}_i$. The difference between the initial positions is $\mathbf{A} = \mathbf{P} - \mathbf{P}^\circ$ and between the final positions $\mathbf{A}' = \mathbf{P}' - \mathbf{P}'^\circ$. The difference between the differences is then

$$\delta \mathbf{A} = \mathbf{A}' - \mathbf{A} = \mathbf{u} - \mathbf{u}^\circ. \tag{2.1}$$

The displacement of any point in the body is a function of its original position, so that

$$\begin{aligned} u_i^\circ &= u_i^\circ(x_1^\circ, x_2^\circ, x_3^\circ) \\ u_i &= u_i(x_1, x_2, x_3) \end{aligned} \tag{2.2}$$

and by expanding to first order around $(x_1^\circ, x_2^\circ, x_3^\circ)$,

$$\mathrm{d}A_i = u_i - u_i^\circ = \left(u_i^\circ + \frac{\partial u_i}{\partial x_1} A_1 + \frac{\partial u_i}{\partial x_2} A_2 + \frac{\partial u_i}{\partial x_3} A_3 \right) - u_i^\circ = \frac{\partial u_i}{\partial x_j} A_j. \tag{2.3}$$

Then, rewriting Eq. (2.3) in the equivalent form,

$$\mathrm{d}A_i = \left[\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \right] A_j, \tag{2.4}$$

and introducing the symmetric quantity, ε_{ij}, and the skew-symmetric quantity, ω_{ij}, defined by

$$\begin{aligned} \varepsilon_{ij} = \varepsilon_{ji} &\equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\ \omega_{ij} = -\omega_{ji} &\equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \end{aligned} \tag{2.5}$$

Equation (2.4) becomes

$$\delta A_i = \varepsilon_{ij} A_j + \omega_{ij} A_j, \tag{2.6}$$

which can be written in vector-tensor form as[2]

$$\delta \mathbf{A} = \underline{\boldsymbol{\varepsilon}} \mathbf{A} + \underline{\boldsymbol{\omega}} \mathbf{A}, \tag{2.7}$$

or, in matrix form as

$$\begin{aligned}[\delta A] &= [\varepsilon][A] + [\omega][A] \\ &= \begin{bmatrix} \delta A_1 \\ \delta A_2 \\ \delta A_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_{21} & \omega_{13} \\ \omega_{21} & 0 & -\omega_{32} \\ -\omega_{13} & \omega_{32} & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}.\end{aligned} \tag{2.8}$$

The quantities $\partial u_i/\partial x_j$ are termed *distortions*, and as now shown, the $\varepsilon_{ij} A_j$ portion of δA_i represents local straining, while the $\omega_{ij} A_j$ portion represents local rigid body rotation. Therefore, $\underline{\boldsymbol{\varepsilon}}$ is termed the *strain tensor*.

2.2.1.1 Local straining and components of strain

To reveal the effect of the $\underline{\boldsymbol{\varepsilon}}$ tensor on $\mathbf{A}$ to produce the new vector, $\mathbf{A}'$, we write $\mathbf{A}' = \mathbf{A} + \delta\,\mathbf{A}$, where $\delta\, A_i = \varepsilon_{ij} A_j$, so that

$$\begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_{11})A_1 + \varepsilon_{12}A_2 + \varepsilon_{13}A_3 \\ \varepsilon_{12}A_1 + (1+\varepsilon_{22})A_2 + \varepsilon_{23}A_3 \\ \varepsilon_{13}A_1 + \varepsilon_{23}A_2 + (1+\varepsilon_{33})A_3 \end{bmatrix}. \tag{2.9}$$

Then, according to Eq. (2.9), if $\mathbf{A}$ lies along $\hat{\mathbf{e}}_1$, as illustrated in Fig. 2.2a, it will be transformed into the vector $\mathbf{A}' = (1+\varepsilon_{11})A\hat{\mathbf{e}}_1 + \varepsilon_{12}A\hat{\mathbf{e}}_2 + \varepsilon_{13}A\hat{\mathbf{e}}_3$, as shown in Fig. 2.2b. Dropping second-order terms, its length will be increased by $\varepsilon_{11}A$ and it will be sheared in the direction $\hat{\mathbf{e}}_2$ by the distance $\varepsilon_{12}A$, and in the direction $\hat{\mathbf{e}}_3$ by $\varepsilon_{13}A$. Similar results will be obtained when $\mathbf{A}$ lies initially along $\hat{\mathbf{e}}_2$ or $\hat{\mathbf{e}}_3$.

The components ε_{11}, ε_{22}, and ε_{33} are seen to be the fractional extensions of the local medium in the $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ directions, respectively, and are termed *normal strains*. On the other hand, as evident in Fig. 2.2, the quantity ε_{12} is a measure of the extent by which the local material is sheared through the angles $\phi_1 = \phi_2 = \varepsilon_{12}$ in the $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_1$ directions, respectively. It can therefore be expressed in the form $\varepsilon_{12} = (\phi_1 + \phi_2)/2 = \phi_{12}/2$, where ϕ_{12} is the total angle of shear that converts the square cross section in Fig. 2.2 into a parallelogram. Similar results are obtained for ε_{13} and ε_{23}. The quantities ε_{12}, ε_{13}, and ε_{23} are therefore identified as half the

[2] The quantities ε_{ij} and ω_{ij} are the components of the second-rank tensors, $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\omega}}$, respectively. A second-rank tensor possesses nine components and maps one vector into another, as in Eq. (2.7), where $\mathbf{A}$ is linearly transformed into $\delta\,\mathbf{A}$ (Nye, 1957).

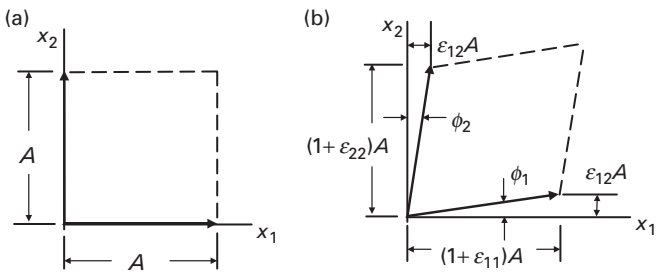

Figure 2.2 Deformation of vector **A**, lying initially along either x_1 or x_2: (a) before strain, (b) after strain.

total shear angles experienced by the local material in the $x_3 = 0$, $x_2 = 0$, and $x_1 = 0$ planes, respectively, and are termed *shear strains*.[3] The local deformation in the immediate vicinity of a point is therefore completely described by six independent components of $\underline{\boldsymbol{\varepsilon}}$, i.e., the three normal strains, ε_{11}, ε_{22}, and ε_{33}, and the three shear strains, $\varepsilon_{12} = \varepsilon_{21}$, $\varepsilon_{13} = \varepsilon_{31}$, and $\varepsilon_{23} = \varepsilon_{32}$.

2.2.1.2 Local rigid body rotation

To reveal the effect of applying the skew-symmetric $\underline{\boldsymbol{\omega}}$ tensor to **A**, as in Eq. (2.8), we consider the general matrix equation that yields the change in **A**, i.e., $\delta\mathbf{A}$, owing to an infinitesimal right-handed rotation of **A** by the angle $\delta\theta$ around an axis parallel to the unit vector $\hat{\boldsymbol{\omega}}$. This can be written as

$$\delta\mathbf{A} = \delta\boldsymbol{\omega} \times \mathbf{A}, \tag{2.10}$$

where $\delta\boldsymbol{\omega}$ is an *infinitesimal rotation vector* given by

$$\delta\boldsymbol{\omega} = \delta\theta\hat{\boldsymbol{\omega}}. \tag{2.11}$$

The vector $\delta\mathbf{A}$ is perpendicular to **A** and, to first order, $(\mathbf{A} + \delta\,\mathbf{A}) \cdot (\mathbf{A} + \delta\,\mathbf{A}) = \mathbf{A} \cdot \mathbf{A}$. Therefore, **A** remains of constant length but is rotated through the angle $\delta\theta = |\,\delta\,\boldsymbol{\omega} \times \mathbf{A}|A^{-1}$. Then, writing out the expression for $\delta\mathbf{A}$ in full,

$$\delta\mathbf{A} = \delta\boldsymbol{\omega} \times \mathbf{A} = (-\delta\omega_3 A_2 + \delta\omega_2 A_3)\hat{\mathbf{e}}_1 + (\delta\omega_3 A_1 - \delta\omega_1 A_3)\hat{\mathbf{e}}_2 + (-\delta\omega_2 A_1 + \delta\omega_1 A_2)\hat{\mathbf{e}}_3 \tag{2.12}$$

or, alternatively,

$$\begin{bmatrix} \delta A_1 \\ \delta A_2 \\ \delta A_3 \end{bmatrix} = \begin{bmatrix} 0 & -\delta\omega_3 & \delta\omega_2 \\ \delta\omega_3 & 0 & -\delta\omega_1 \\ -\delta\omega_2 & \delta\omega_1 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}. \tag{2.13}$$

[3] The shear strain, ε_{ij} $(i \neq j)$, employed in this book, is a component of the strain tensor, $\underline{\boldsymbol{\varepsilon}}$, and is equal to half the "engineering shear strain," which is often employed in the literature (e.g., Timoshenko and Goodier, 1970) and is not the component of a tensor.

The rotation matrix in Eq. (2.13) and the $[\omega]$ matrix in Eq. (2.8), where

$$\delta\omega_1 = \omega_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \quad \delta\omega_2 = \omega_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)$$
$$\delta\omega_3 = \omega_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{2.14}$$

are seen to have the same form, thus confirming that the latter matrix indeed represents rigid body rotation.

2.2.2 Relationships for strain components

2.2.2.1 Transformation of strain components due to rotation of coordinate system

When a strain tensor is known in a given coordinate system it is often necessary to find an expression for the same strain when it is referred to a new coordinate system rotated with respect to the original system. This can be accomplished after first finding the relationship between a given vector displacement referred to the old system and then to the new system.

Let $\mathbf{u} = u_i\hat{\mathbf{e}}_i$ and $\mathbf{u}' = u_i'\hat{\mathbf{e}}_i'$ represent the same displacement vector referred to the old and new coordinate systems, respectively. The components of the vector in the new system in terms of its components in the old system are then

$$u_i' = (u_1\hat{\mathbf{e}}_1 + u_2\hat{\mathbf{e}}_2 + u_3\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_i' = u_j(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_i') = l_{ij}u_j \text{ or } [u'] = [l][u], \tag{2.15}$$

where l_{ij} is the cosine of the angle between $\hat{\mathbf{e}}_i'$ and $\hat{\mathbf{e}}_j$. Conversely, the old components in terms of the new components are given by

$$u_i = (u_1'\hat{\mathbf{e}}_1' + u_2'\hat{\mathbf{e}}_2' + u_3'\hat{\mathbf{e}}_3')\cdot\hat{\mathbf{e}}_i = u_j'(\hat{\mathbf{e}}_j'\cdot\hat{\mathbf{e}}_i) = l_{ji}u' \text{ or } [u] = [l]^{\mathrm{T}}[u']. \tag{2.16}$$

Solving Eq. (2.15) for $[u]$,

$$[u] = [l]^{-1}[u'] \tag{2.17}$$

and, by comparing Eqs. (2.16) and (2.17),

$$[l]^{\mathrm{T}} = [l]^{-1}. \tag{2.18}$$

Therefore,

$$[l][l]^{\mathrm{T}} = [l]^{\mathrm{T}}[l] = [\mathrm{I}]. \tag{2.19}$$

Every column vector and row vector in $[l]$ is a unit vector, and every pair of column vectors and every pair of row vectors is orthogonal. Therefore, $[l]$ is termed a *unitary orthogonal matrix*.

The deformation of the vector **A** in the old coordinate system and in the new system by the strain tensor will be of the respective forms

$$[\delta A] = [\varepsilon][A] \qquad [\delta A'] = [\varepsilon'][A']\,. \tag{2.20}$$

The transformation matrix $[l]$ for the vector **u** in Eq. (2.15) will also apply to the vectors $\delta\,\mathbf{A}'$ and $\mathbf{A}'$ in Eq. (2.20), and therefore

$$\begin{aligned} [\delta A'] &= [\varepsilon'][A'] \\ [l][\delta A] &= [\varepsilon'][l][A] \\ [\delta A] &= [l]^{\mathrm{T}}[\varepsilon'][l][A]. \end{aligned} \tag{2.21}$$

Then, comparing this result with Eq. (2.20), the strain tensors referred to the two systems are related by

$$[\varepsilon] = [l]^{\mathrm{T}}[\varepsilon'][l] \tag{2.22}$$

and, by inverting Eq. (2.22) by use of Eq. (2.18),

$$[\varepsilon'] = [l][\varepsilon][l]^{\mathrm{T}}. \tag{2.23}$$

The transformations given by Eqs. (2.22) and (2.23) may also be expressed in the component forms:

$$\varepsilon'_{ij} = l_{im}l_{jn}\varepsilon_{mn}, \tag{2.24}$$

$$\varepsilon_{ij} = l_{mi}l_{nj}\varepsilon'_{mn}. \tag{2.25}$$

All second-rank tensors follow these transformation laws.

2.2.2.2 Principal coordinate system for strain tensor

Using these results it is now shown that for any state of strain it is always possible to find a coordinate system, termed the *principal coordinate system*, that causes the strain tensor to take the simple diagonal matrix form[4]

$$[\tilde{\varepsilon}] = \begin{bmatrix} \tilde{\varepsilon}_{11} & 0 & 0 \\ 0 & \tilde{\varepsilon}_{22} & 0 \\ 0 & 0 & \tilde{\varepsilon}_{22} \end{bmatrix}, \tag{2.26}$$

where the diagonal elements are known as the *principal strains*. When the principal coordinate system is employed, and the strain tensor, in the form of Eq. (2.26), is applied to various vectors in the medium, it is readily seen that a vector lying along any one of the three coordinate axes (i.e., the three *principal directions*) remains non-rotated and simply undergoes a fractional change in length corresponding to the principal strain along that axis. The principal directions are therefore special directions in which vectors embedded in the medium are simply changed in length

[4] All quantities referred to a principal coordinate system in this section are distinguished by a tilde, as in Eq. (2.26).