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Peeking at partizan misère quotients

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1. Introduction

In two-player combinatorial games, the last player to move either wins (normal play) or loses (misère play). Traditionally, normal play games have garnered more attention due to the group structure which arises on such games. Less work has been done with games played under the misère play convention, Just as in normal play, misère games can be placed in equivalence classes, where two games G and H are equivalent if the outcome class of $G+K$ is the same as the outcome class of $H+K$ for all games K. However, Conway showed that, unlike in normal play, these misère equivalence classes are sparsely populated, making the analysis of misère games under such equivalence classes far less useful than their normal play counterparts [ONAG]. Even though these equivalence classes are sparse, Conway developed a method, called genus theory, for analyzing impartial games played under the misère play convention [Allen 2006; WW; ONAG]. For years, this was the only universal tool available for those studying misère games.

In [Plambeck 2009; 2005; Plambeck and Siegel 2008; Siegel 2006; 2015b], many results regarding impartial misère games have been achieved. These results were obtained by taking a game, restricting the universe in which that game was played, and obtaining its misère quotient. However, while, as Siegel [2015a] says "a partizan generalization exists", few results have been obtained regarding the structure of the misère quotients which arise from partizan games.

For a game $G=\{G^L \mid G^R\}$, we define $\overline{G}=\{\overline{G^R} \mid \overline{G^L}\}$. Those familiar with normal play will notice that under the normal play convention rather than $\overline{G}$, we would generally write $-G$. In normal play, this nomenclature is quite sensible as $G+(-G)=0$ [Albert et al. 2007], giving us the Tweedledum–Tweedledee principle; the second player can always win the game $G+(-G)$ by mimicking the move of the first player, but in the other component. However, in misère play, not only does the Tweedledum–Tweedledee strategy often fail, $G+\overline{G}$ is not necessarily equivalent to 0. For example, $*_2+\overline{*_2}=*_2+*_2$ is not equivalent to 0 [Allen 2006; WW]. However, having the property that $G+\overline{G}$ is equivalent to

0 is much desired, as it gives a link to which partizan misère games may behave like their normal counterparts.

To this end, this paper shows that

(1) $*+*$ is indistinguishable from 0 in the universe of all-small games, and

(2) there exists a set of games with the property that $G+\overline{G}$ is always equivalent to 0 relative to all-small games.

Using these results, the misère quotients of two nontrivial partizan examples are calculated. One such example has cardinality nine, a cardinality not found within impartial misère quotients [Plambeck and Siegel 2008]. As well, the partially ordered outcome set of one this example is given. This paper concludes with a list of six open problems of varying depth and scope in the area of partizan misère quotients.

While some elementary definitions are reviewed, this paper assumes the reader has a basic familiarity with the impartial misère quotient construction developed by Plambeck and Siegel.

2. Indistinguishability

This section contains a brief review of the indistinguishability definitions developed by Plambeck and Siegel.

Let G be a game (impartial or partizan). Then we use $o^-(G)$ to denote the misère play outcome of G, keeping the minus sign so as to not forget that we are dealing with misère play games rather than normal play ones.

We say that a set of games Υ is *closed* if it is

(1) closed under addition, i.e., if $G, H \in \Upsilon$, then $G+H \in \Upsilon$, and

(2) option closed, i.e., if $G \in \Upsilon$, then every option of G is also in Υ.

Frequently, the set of games over which we want to work is not closed. As such, we are required to take the *closure of the set* where for Υ a set of games, $c\ell(\Upsilon)$ is the smallest closed set such that $\Upsilon \subseteq c\ell(\Upsilon)$.

Suppose Υ to be a closed set of games with $G, H \in \Upsilon$. Then *G and H are indistinguishable over* Υ if

$$o^-(G+K) = o^-(H+K) \text{ for all } K \in \Upsilon,$$

and we write

$$G \equiv H \pmod{\Upsilon}.$$

Indistinguishability $\pmod{\Upsilon}$ is both an equivalence relation compatible with addition, and so, $\Upsilon/_{\equiv_\Upsilon}$ is well-defined and forms a monoid [Plambeck and Siegel 2008], which is the *misère quotient of* Υ. We denote this monoid by

$\mathscr{Q}(\Upsilon)$. Moreover, $\mathscr{Q}(\Upsilon)$ is partitioned into four disjoint outcome sets, $\mathscr{N}$, $\mathscr{P}$, $\mathscr{L}$, and $\mathscr{R}$, meaning Next, Previous, Left, and Right respectively, where, for example, $[G]_{\equiv_\Upsilon} \in \mathscr{N}$ if and only if $o^-(G) = \mathscr{N}$.

For a more detailed discussion of misère quotients, their development, and results on the monoid structures obtained, this paper refers the reader to the work of Plambeck and Siegel, most notably [Plambeck 2009; 2008; Plambeck 2005; Siegel 2015b; 2006].

3. All-small games and $* + *$

Suppose that Υ is a closed set of *impartial games* with $* \in \Upsilon$. Then we have the following result:

Proposition 3.1. $* + * \equiv 0 \pmod{\Upsilon}$ [WW].

However, if Υ contains certain partizan games, Proposition 3.1 fails.

Proposition 3.2. *Let* $1 = \{0|\cdot\}$ *and suppose* $1, * \in \Upsilon$, *a closed set of games. Then* $* + * \not\equiv 0 \pmod{\Upsilon}$.

Proof. It is easy to show that while $o^-(1) = \mathscr{R}$, Left can force a win if Right moves first in $1 + * + *$. □

Thus, while we cannot extend Proposition 3.1 to all partizan games, we can extend the result to all-small games, as shown in the following theorem.

Theorem 3.3. *Let* Υ *be a closed set of all-small games with* $* \in \Upsilon$. *Then* $* + * \equiv 0 \pmod{\Upsilon}$.

The proof of this theorem extends that of a similar result for impartial games given in [Siegel 2006].

Proof. Take $G \in \Upsilon$. We want $o^-(G + * + *) = o^-(G)$. Proceed by induction on the options of G.

We know that $o^-(0) = \mathscr{N}$, and $o^-(* + *) = \mathscr{N}$, so the base case holds.

Now suppose true for all options of G and consider G.

Since G is nonzero and all-small, Left must have a move from G. Suppose Left wins moving first in G. Then Left wins moving first in $G + * + *$ by moving to $G^L + * + *$, where G^L is a winning position for Left moving second. Since G^L is an option of G, by induction, $o^-(G^L + * + *) = o^-(G^L)$. Therefore Left wins moving second in $G^L + * + *$, and so Left wins moving first in $G + * + *$.

Suppose Left wins moving second in G. Right has two possible starting moves in $G + * + *$. Ge may move to either $G^R + * + *$ or to $G + *$. Suppose Right moves to $G^R + * + *$. By induction, $o^-(G^R + * + *) = o^-(G^R)$, where Left has a winning move moving first in G^R. Thus Left has a winning move moving first in $G^R + * + *$, and so, this is not a good opening move for Right. If Right moves to $G + *$, then Left responds with G, leaving Right to make the next move in G, and so Left wins.

Therefore, if Left moving first (or second) wins G, then Left moving first (or second) wins $G + * + *$. A symmetric argument works for Right.

Therefore $o^-(G) = o^-(G + * + *)$, and so $* + * \equiv 0 \pmod{\Upsilon}$ when Υ contains only all-small games. □

Corollary 3.4. *Let Υ be the set of all all-small games. Then $* + * \equiv 0 \pmod{\Upsilon}$.*

Proof. The set of all all-small games is closed. Thus the result follows from Theorem 3.3. □

The importance of this result is two-fold. Not only does it extend a result for impartial games, it also allows us to reduce misère monoid calculations when examining closed sets of all-small games, as we need only consider positions which contain at most one $*$.

4. Conjugation and equivalence with 0

As reviewed in the introduction, $G + \overline{G}$ is not necessarily equivalent to 0 for G played under the misère convention. However, this does raise an interesting area of investigation. For what G is it true that $G + \overline{G} \equiv 0 \pmod{c\ell(G, \overline{G})}$? This section gives an infinite set of games for which this is true.

Definition 4.1. Let G be a game. Then G is a *binary game* if at any point, a player has either no moves available or exactly one move available.

Definition 4.2. A position G is called ab$n$ if

(1) G is all-small,

(2) G is binary,

(3) each alternating path in the game tree of G is of length n or less.

Consider the games given in Figure 1. Then $*$ is ab1, G_1, G_2, and G_3 are ab2, and H_1 and H_2 are ab4.

Note that if G is ab$n$, then G is ab$m$ for all $m > n$. Also note that if G is ab$n$, then all of G's options are also ab$n$.

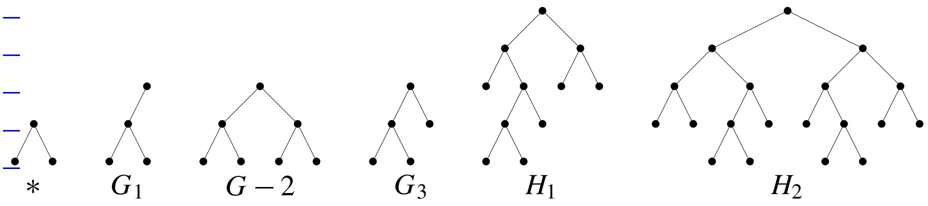

Figure 1. The games $*$, G_1, G_2, G_3, H_1, and H_2.

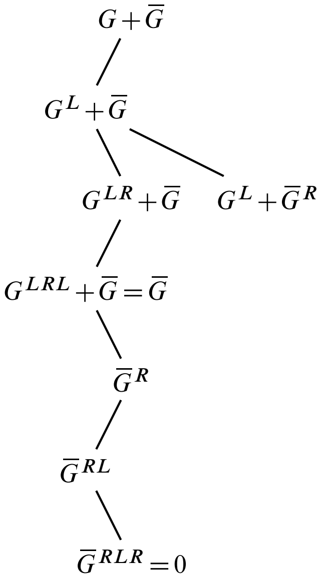

Figure 2. Left wins $G+\overline{G}$ moving first if G is ab3 and $G^{LRL}=0$.

We restrict ourselves first to examining games which are ab3. We will show that if G is ab3, then $G+\overline{G}\equiv 0 \pmod{\mathrm{c}\ell(G,\overline{G})}$. We first require the following proposition.

Proposition 4.3. *Let G be* ab3*. Then* $o^-(G+\overline{G})=\mathcal{N}$.

Proof. Proceed by induction on the birthday of G.

Suppose $G=0$. Then $o^-(0+0)=o^-(0)=\mathcal{N}$, as required.

Suppose true for all K which are ab3 and which have smaller birthday than G. Consider G.

Suppose $G^L=0$. Then $\overline{G}^R=0$. Left moves first in $G+\overline{G}$ to $G^L+\overline{G}=\overline{G}$. Right's only response is to $\overline{G}^R=0$, and so Left wins.

Now suppose $G^{LRL}=0$. Then $\overline{G}^{RLR}=0$. Figure 2 shows how Left moving first can win $G+\overline{G}$, noting that $\overline{G^L}=\overline{G}^R$, and that the birthday of G^L is strictly less than the birthday of G, so $o^-(G^L+\overline{G}^R)=\mathcal{N}$ by induction.

Suppose that $G^{LR}=0$. Then $\overline{G}^{RL}=0$. If $G^R=0$ or $G^{RLR}=0$, then repeat one of the above arguments to get that Left wins moving first in $G+\overline{G}$. Otherwise, suppose that $G^{RL}=0$. Figure 3 shows how Left moving first can win $G+\overline{G}$.

A symmetric argument shows how Right wins moving first in $G+\overline{G}$, and so the result holds. □

We can now prove our main result.

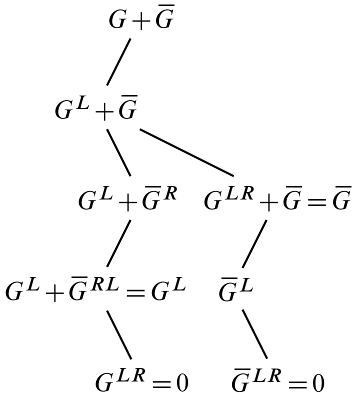

Figure 3. Left wins $G+\overline{G}$ moving first if G is ab3 and $G^{LR}=G^{RL}=0$.

Theorem 4.4. *Suppose G is* ab3 *and let Υ be the set of all all-small games. Then* $G+\overline{G}\equiv 0 \pmod{\Upsilon}$.

Proof. This proof is very similar to that of Theorem 3.3.

Proceed by induction on the birthday of G.

Suppose $G=0$. Then clearly $0+0\equiv 0 \pmod{\Upsilon}$.

Suppose true for all L which are ab3 and which have smaller birthdays than G. Consider $G+\overline{G}$. We want for any all-small H, $o^-(H)=o^-(H+G+\overline{G})$. Proceed by induction on H.

Suppose $H=0$. Then $o^-(0)=\mathscr{N}$, and, by Proposition 4.3, $o^-(G+\overline{G})=\mathscr{N}$. This shows the base case for the induction on H.

Now suppose $o^-(K)=o^-(K+G+\overline{G})$ for all all-small K with lesser birthday than that of H.

Since H is all-small and nonzero, we know that some H^L must exist. Suppose Left moving to H^L is a winning move for Left moving first in H. Claim that Left can win moving first in $H+G+\overline{G}$ with the move $H^L+G+\overline{G}$. Since the birthday of H^L is strictly less than the birthday of H, have $o^-(H^L+G+\overline{G})=o^-(H^L)$. Since Left wins moving first in H, this means $o^-(H)=\mathscr{P}$ or $\mathscr{L}$, so $o^-(H^L+G+\overline{G})=\mathscr{N}$ or $\mathscr{L}$, so Left wins moving first in $H+G+\overline{G}$.

Suppose Left wins moving second in H. Consider Right's three possible first moves in $H+G+\overline{G}$, given in Figure 4. Suppose Right makes the first move to $H^R+G+\overline{G}$. Since Left wins moving second in H, this gives $o^-(H^R)=\mathscr{N}$ or $\mathscr{L}$. Since the birthday of H^R is strictly less than the birthday of H, by induction, $o^-(H^R+G+\overline{G})=\mathscr{N}$ or $\mathscr{L}$, so Left wins $H+G+\overline{G}$ moving second if Right's first move is to $H^R+G+\overline{G}$.

Suppose Right makes the first move to $H+G^R+\overline{G}$. Left responds by moving to $H+G^R+\overline{G}^L$. Since $\overline{G^R}=\overline{G}^L$ and G^R is ab3, by induction,

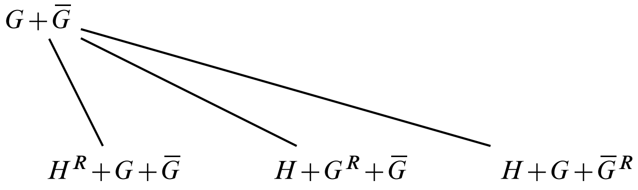

Figure 4. Right's possible opening moves in $H+G+\overline{G}$.

$o^-(H+G^R+\overline{G}^L)=o^-(H)$. Since Left wins moving second in H, by induction, Left wins moving second in $H+G^R+G$. Similarly, if Right's first move is to $H+G+\overline{G}^R$, then Left will also win.

Therefore, if Left moving first (or second) wins H, then Left moving first (or second) wins $H+G+\overline{G}$. A symmetric argument works for Right.

Therefore $o^-(H)=o^-(H+G+\overline{G})$, and so $G+\overline{G}\equiv 0 \pmod{\Upsilon}$. □

Corollary 4.5. *Let Υ be a closed set of all-small games, not necessarily all all-small games. Suppose G is* ab3*. Then $G+Gb\equiv 0 \pmod{c\ell(\Upsilon)}$.*

Proposition 4.3, Theorem 4.4, and Corollary 4.5 are surprising results. The proposition gives pairs of games $(G,\overline{G})$ in which we always know the outcome class of their sum under the misère play convention. Other than for tame games, impartial games whose misère quotients are the same as that of sums of Nim heaps, very little is known about how to deal with disjunctive sum of misère games. See [Allen 2006] and [ONAG] for a discussion on the sums of tame games, and [Mesdal and Ottaway 2007] for a discussion on some difficulties arising with the disjunctive sum on arbitrary misère games. Theorem 4.4 parallels the result discussed at the beginning of this section, namely that under normal play, $G+(-G)$ always equals 0.

A natural question is how far can Theorem 4.4 be extended? Some simple leg work shows that, for H_2 given in Figure 1, $o^-(H_2+\overline{H_2})=\mathscr{P}$, and so $H_2+\overline{H_2}\not\equiv 0 \pmod{c\ell(H_2,\overline{H}_2)}$. Hence, Theorem 4.4 does not extend to all ab$n$ games for $n\geq 4$. However, it would still be worth investigating for which ab$n$ games with $n\geq 4$ have the result given in Theorem 4.4.

5. Two examples of partizan misère quotients

Consider the game $\downarrow=\{*\mid 0\}$. In this section, we will calculate $\mathscr{Q}(c\ell(\downarrow))$ and $\mathscr{Q}(c\ell(\downarrow,\overline{\downarrow}))$.

5.1. ***The partizan misère quotient of $c\ell(\downarrow)$.*** The positions in $c\ell(\downarrow)$ are 0, $*$, and $\downarrow$. Since $\downarrow$ is all-small, Theorem 3.3 gives that $*+*\equiv 0 \pmod{c\ell(\downarrow)}$. Thus, every position in $c\ell(\downarrow)$ is indistinguishable from one of the form $m\downarrow$ or $*+m\downarrow$, where $m\downarrow$ denotes the disjunctive sum of m copies of $\downarrow$. A bit of work [Allen 2009] gives the following for the outcome classes of the positions $n{*}+m\downarrow$:

	$m=0$	$m=1$	$m=2$	$m=3$	$m\geq 4$
$n\equiv 0$	$\mathscr{N}$	$\mathscr{L}$	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$
$n\equiv 1$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{R}$

Moreover, we can see that

(1) $4\downarrow\equiv u\downarrow \pmod{c\ell(\downarrow)}$ for any $u\geq 4$,

(2) $4\downarrow\equiv *+u\downarrow \pmod{c\ell(\downarrow)}$ for any $u\geq 4$.

Enumerating the elements then gives us:

$$0,\quad *,\quad \downarrow,\quad 2\downarrow,\quad 3\downarrow,\quad 4\downarrow,\quad *+\downarrow,\quad *+2\downarrow,\quad *+3\downarrow,$$

all of which are pairwise distinguishable. Note that if $o^-(G)\neq o^-(H)$, then the two elements are distinguished by 0. Table 1 shows the distinguishing elements in $c\ell(\downarrow)$ when $o^-(G)=o^-(H)$.

With the mappings

$$0\mapsto 1,\quad *\mapsto a,\quad \downarrow\mapsto d,$$

the following monoid is achieved:

$$\begin{aligned}\mathscr{Q}(c\ell(\downarrow)) &= \langle 1,a,d \mid a^2=1,\ d^4=d^5=ad^4\rangle,\\ \mathscr{N} &= \{1,ad,ad^2,ad^3\},\\ \mathscr{P} &= \{a,d^2\},\\ \mathscr{L} &= \{d\},\\ \mathscr{R} &= \{d^3,d^4\},\end{aligned}$$

Position 1	Position 2	distinguishing element
0	$*+\downarrow$	$*$
0	$*+2\downarrow$	$\downarrow$
0	$*+3\downarrow$	$\downarrow$
$*+\downarrow$	$*+2\downarrow$	$*$
$*+\downarrow$	$*+3\downarrow$	$*$
$*+2\downarrow$	$*+3\downarrow$	$*$
$*$	$2\downarrow$	$2\downarrow$
$3\downarrow$	$4\downarrow$	$*$

Table 1. Positions of $c\ell(\downarrow)$ and the elements which distinguish them.

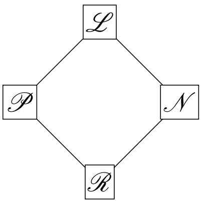

Figure 5. Outcome class partial order.

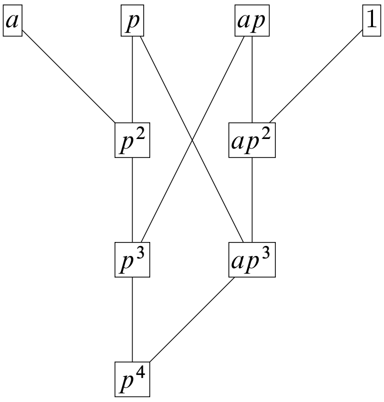

Figure 6. The partially ordered set of $\mathscr{Q}(\mathrm{c}\ell\,(\downarrow))$.

with the additive notation in $\mathrm{c}\ell\,(\downarrow)$ becoming multiplicative notation in $\mathscr{Q}(\mathrm{c}\ell\,(\downarrow))$.

This example demonstrates an important difference between partizan and impartial misère quotients. In impartial games, every finite indistinguishability quotient has either cardinality one or is of even cardinality [Plambeck and Siegel 2008]. Contrast this with the cardinality of $\mathscr{Q}(\mathrm{c}\ell\,(\downarrow))$, which is nine.

Another possible area for investigation regarding partizan misère quotients is on the partial order of the elements. Recall that, under monoid multiplicative notation,

$$x \geq y \ \text{ if } \ o^-(xz) \geq o^-(yz) \qquad \text{for all monoid elements } z,$$

and that, in terms of outcomes, the outcome lattice is given in Figure 5.

The partially ordered set of $\mathscr{Q}(\mathrm{c}\ell\,(\downarrow))$ is given in Figure 6. However, while these sets can be calculated, no general results on such partially ordered sets have been obtained.

5.2. ***The partizan misère quotient of $\mathrm{c}\ell\,(\downarrow, \uparrow)$.*** Note that $\overline{\downarrow} = \{0 \mid *\} = \uparrow$. Thus $\mathscr{Q}(\mathrm{c}\ell(\downarrow, \overline{\downarrow})) = \mathscr{Q}(\mathrm{c}\ell\,(\downarrow, \uparrow))$.

We will now calculate $\mathscr{Q}(\mathrm{c}\ell(\downarrow,\uparrow))$. Since $\mathrm{c}\ell(\downarrow,\uparrow)$ is a set of all-small games, we can apply Theorem 3.3. Since $\downarrow$ is ab2, and hence ab3, we can apply Corollary 4.5. Combining these two results, we get that all positions in $\mathrm{c}\ell(\downarrow,\uparrow)$ are indistinguishable from one of the following:

$$0,\quad *,\quad m\downarrow,\quad \ell\uparrow,\quad *+m\downarrow,\quad \text{or}\quad *+\ell\uparrow.$$

Moreover, it can be shown that any two positions of the above form are distinguishable ([Allen 2009]).

With the mappings

$$0\mapsto 1,\quad a\mapsto a,\quad \downarrow\mapsto d,\quad \uparrow\mapsto u,$$

the following monoid is achieved:

$$\mathscr{Q}(\mathrm{c}\ell(\downarrow,\uparrow)) = \left\langle 1, a, d, u \;\middle|\; a^2=1,\, d^m u^n = \begin{cases} d^{m-n} & \text{if } m>n, \\ u^{n-m} & \text{if } m\le n, \end{cases}\right\rangle,$$
$$\mathscr{N} = \{1, ad, ad^2, ad^3, au, au^2, au^3\},$$
$$\mathscr{P} = \{a, d^2, u^2\},$$
$$\mathscr{L} = \{d, u^3, u^4, u^5, \ldots, au^4, au^5, au^6 \cdots\},$$
$$\mathscr{R} = \{u, d^3, d^4, d^5, \ldots, ad^4, ad^5, ad^6, \ldots\},$$

with the additive notation in $\mathrm{c}\ell(\downarrow,\uparrow)$ having become multiplicative notation in $\mathscr{Q}(\mathrm{c}\ell(\downarrow,\uparrow))$ (the outcome class calculations can be seen in [Allen 2009]). Notice that $\mathscr{Q}(\mathrm{c}\ell(\downarrow,\uparrow))$ is an infinite monoid. Thus, as in the impartial case, infinite misère monoids exist.

The partially ordered set of this monoid is particularly unpleasant, but is calculated in full in [Allen 2009].

It should also be noted that if we consider the misère monoid simply as a monoid without the outcome tetrapartitions, than it is isomorphic to the group $\mathbb{Z}_2 \oplus \mathbb{Z}$, just as in normal play.

6. Conclusion

Theorems 3.3 and 4.4 give a good starting base for further investigation of partizan misère quotients. While, as Theorem 3.3 shows, all-small games share some results with impartial games, the misère quotient of $\mathrm{c}\ell(\downarrow)$, which has nine elements, shows that even restricting ourselves to all-small games can yield results which do not appear for impartial misère quotients. This paper concludes with some possibilities for further research in the area of partizan misère quotients: