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Brian Koberlein and David Meisel
Excerpt
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1 Introduction

Astrophysics draws upon a wide range of topics in astronomy and physics. Topics as widely ranging as observational techniques, thermodynamics, and general relativity are all central to the material covered in this text. Many readers, particularly undergraduate students, will have only passing experience with some of these foundational concepts. It is impossible for us to provide a comprehensive review of these within the scope of this text, but we begin with a basic overview of the most central topics necessary to approach the astrophysical subjects to be covered. In later chapters we build on these various terms in greater detail. Although we do assume more than a general background in physics and astronomy, we do not provide a comprehensive discussion of the basics, as these can be obtained elsewhere. Instead we present only those more advanced background concepts as needed to the task at hand.

1.1 Fundamental stellar properties

One of the central goals of astronomy is the specification of the properties of the sun, stars, and other self-luminous bodies in the universe. Learning about the ranges of these properties and how the quantities characterizing them are determined are a major part of any astronomy course, so we only briefly discuss them here.

1.1.1 Cosmic distance scales

The most fundamental property of a celestial object is its distance from another body. Usually the reference is the sun, the star nearest to Earth. Within the solar system, distances are specified using *astronomical units* (AU), equal to the mean distance between Earth and sun. Outside the solar system, distances are so large that one immediately switches to another unit, the *parsec*. The parsec was devised when distances to stars were first measured by the method of trigonometric parallax, and distance was determined from the annual shift of an object's angular position in the sky.[1]

At the time of Kepler and Copernicus, the difficulty in getting stellar distances was that the parallax angle was very small. Most astronomers of the time reasoned that the brightest stars would be the closest, and so early measurements focused on the brightest stars. But that assumption works only if all stars are about the same brightness, which is not the case. The brightest appearing stars most often also have the greatest luminous output, and therefore they can be seen over even greater distances than stars such as the sun. Hence the parallax searches using the brightest stars largely failed. When it was possible to measure parallaxes, the angles were always smaller than 1 second of arc (arcsecond) or 4.85×10^{-6} radians.

[1] The symbol π was often used for parallax angle, which is easy to confuse with the number π.

The distance unit for parsec is defined as 206, 265 AU, such that it is the distance at which parallax is 1 arcsecond. The practical parallax limit in modern times is on the order of a milliarcsecond, so the distance limit is about 1 kiloparsec. Objects have been measured at megaparsecs, but this uses other methods, as we shall see. At so-called cosmological distances, one relies on relative brightness to estimate distances.

1.1.2 Cosmic brightness scale

Because the earliest observations of stars were made with the unaided eye and date back beyond the time of ancient Greeks, the brightness is rarely specified using energy units. Instead a unit called *magnitude* is used. When referring to what the human eye would perceive, the quantity called *apparent magnitude*, m, is given.

In modern times magnitudes are often given outside the range of human vision, so a color or wavelength range is specified, such as m_{blue} for a magnitude estimated under a blue filter, or m_{red} for one estimated through a red filter. Magnitudes have some peculiar properties relative to energy measurements that have become standard for historical reasons. The Greeks had only integer magnitudes without 0. Their ranking was such that the unity symbol had the highest value (think first prize) and the largest number had the lowest ranking. Because stars seemed to fall visually on this scale from 1 to 6, the magnitude scale difference in the visual range was 5. Later in history, it was determined that the total magnitude difference of visible stars corresponded to a brightness (luminous energy) ratio of about 100. Thus magnitudes are logarithmic measures of brightness, just as decibels measure audio loudness.

Decibels are base 10 logarithms, but magnitudes are not. Because a difference of $\Delta m = 5$ is a brightness ratio of 100, we have

$$100 = \left(\log_x\right)^5 , \quad (1.1)$$

which yields a logarithmic base of 2.512.[2] This is typically expressed as a base 10 logarithm with a multiplying constant so that the brightness ratio (intensity ratio) I/I_0 becomes

$$\frac{I}{I_0} = 10^{0.4\Delta m}, \quad (1.2)$$

or

$$\Delta m = 2.5 \log_{10}\left(\frac{I}{I_0}\right). \quad (1.3)$$

The apparent magnitude itself does not give any indication of the actual brightness of an object because the intrinsic brightness is attenuated by the inverse square law of distance,

$$I_1 = I_0 \frac{r_1^2}{r_0^2}. \quad (1.4)$$

In astronomy, the standard distance for brightness measurement is not 1 parsec as you might expect, but rather 10 parsecs. The reasons for this are again found in early observations. There are no major stellar bodies within 1 parsec of the sun, but there are many within 10 parsecs, so this was chosen. The *absolute magnitude*, M, is thus defined as the apparent magnitude a star would have at a distance of 10 parsecs. The absolute magnitude can be calculated from the apparent magnitude by

$$M = m - 5\left(\log_{10} D - 1\right), \quad (1.5)$$

where D is the object's distance in parsecs.

[2] See **1-1.Stellar** for details.

1.1.3 Color index and temperature

The color of an object can be specified approximately by the *color index* (C.I.) such that

$$C.I. = m_{\mathrm{blue}} - m_{\mathrm{red}}. \tag{1.6}$$

The color index is related to the color temperature of the object. If it can be assumed that the object is a blackbody, then the Planck function is used to estimate this quantitatively.

The energy density u for a photon gas is defined by the Planck function with the energy in joules,

$$u(\omega, T) = \frac{\hbar\omega^3/\pi^2c^3}{e^{\hbar\omega/kT} - 1}, \tag{1.7}$$

where ω is the angular frequency observed and T is the blackbody temperature. The theoretical color index is then

$$C.I. = 2.5\log_{10}\left(\frac{u(\omega_r, T)}{u(\omega_b, T)}\right). \tag{1.8}$$

In optical astronomy the wavelength form of Planck's radiation law is often used. Thus

$$u(\lambda, T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1}. \tag{1.9}$$

The "blue" minus "red" is a bit of an exaggeration. Usually color indices span smaller wavelength ranges. Johnson and Morgan (1953) introduced a standardized system known as the UBV system. This system used color filters for observing magnitudes in the ultraviolet, blue, and "visible" ranges, from which one could generate $U - B$ and $B - V$ color indices. Modern astronomers use UBVRI standard with mean wavelengths of $U = 3600$ Å, $B = 4400$ Å, $V = 5500$ Å, $R = 7000$ Å, and $I = 9000$ Å.

By convention, 10 000 K is the temperature for which the color index is supposed to be 0 regardless of the wavelength difference. In practical terms, this means there is a correction constant that must be added to the preceding expressions to get the correct color index.[3]

1.1.4 Radius, temperature, and luminosity

Most stellar objects are excellent approximations to spheres, and as such their brightness properties are straightforward. Each luminous object radiates through its surface via the Planck law. To obtain the total energy over all wavelengths, one must integrate the Planck law in explicit form,

$$L = \int u(\lambda, T)\, \mathrm{d}A\mathrm{d}\Omega\mathrm{d}\lambda = 4\pi R^2\sigma T^4, \tag{1.10}$$

known as the Stefan–Boltzmann equation.

The absolute magnitude obtained with the entire Stefan–Boltzmann equation is called the *bolometric* absolute magnitude,

$$M_{bol} = -2.5\log_{10} L. \tag{1.11}$$

For convenience this is often normalized to the values of the sun. Thus the radius of the sun ($R = 1$) and its temperature (5800 K) are assigned along with its absolute magnitude (4.8) to calculate the bolometric absolute magnitude function

$$M_{bol} - M_{\odot} = -2.5\log_{10}\left(\frac{L}{L_{\odot}}\right). \tag{1.12}$$

[3] See **1-1Stellar** for examples.

Because observations are made with color band filters, it is the visual absolute magnitude that is most often determined. Because luminosity is defined over the whole spectrum, we need to correct M_v to obtain M_{bol}. The magnitude correction that is added is called the bolometric correction, and it is the magnitude equivalent of the ratio of the full spectrum to the partial spectrum.

1.2 Determination of stellar mass

Mass is the source of gravitational fields throughout the cosmos, and everything that has mass contributes to this field. In general relativity mass curves space–time, and this is where gravitational fields originate. Even on a less sophisticated scale humans intuitively, but perhaps qualitatively, know the effects of gravity and how moving masses are influenced by it. Because gravity is such a long-range force, its effects are felt across immense distances; therefore its importance within astronomy and astrophysics is paramount.

The determination of mass from orbital mechanics is a very classical pursuit and works surprisingly well as long as the orbiting object is not too close to its main body. It started when Newton re-derived Kepler's laws of motion based on the rules of Newtonian gravity, with the result that the new Kepler's laws had gravitational theory in terms of the mass of one or both binary bodies. If observational factors were favorable, the individual masses could be estimated. When objects are not orbiting, the main way to tell the mass of an object is to observe its perturbations by other masses or to observe perturbations of other masses on it.

For a binary star, the orbit in space is, at least without the presence of disturbing objects, just the usual two-body ellipse. Kepler's laws for the system can be expressed as

$$(m_1 + m_2)\,P^2 = R^3, \tag{1.13}$$

where P is the orbital period in years and $R = a_1 + a_2$ is the sum of the semi-major axes of the two stars in AU. From the center of mass,

$$m_1 a_1 = m_2 a_2. \tag{1.14}$$

These two equations can then be used to determine the masses m_1 and m_2 in solar mass units quite simply. Obtaining values for a_1 and a_2 is a much bigger challenge.

1.2.1 Visual binaries

Just as Galileo discovered moons orbiting Jupiter, later astronomers discovered that some of the stars that appeared as multiple through telescopes were in fact orbiting each other. Early measurements were often just "sketched" relative to points plotted on graph paper. The properties were then measured from the graph paper directly. Later visual measurements were made with a filar micrometer in a polar coordinate system centered on the primary star, or they were provided by photographic means. Even done visually with a micrometer, measurements were liable to much error because the distances between the stars were so small. In modern times, binary positions are measured using speckle interferometers with considerably more precision.

As observed from Earth, the orbit of a binary star around the primary star is an ellipse, but its semi-major axis length and foci positions are distorted. To find the orbital parameters we follow the derivations and conventions given by Smart (1960) based on an earlier method attributed to Kowalsky, as they represent the mid-20th century state of binary orbit determination. At that time, observations were heavily dependent on filar micrometers that produced polar measurements (ρ, θ) of the position of the secondary star relative to the primary star. Here ρ is the radial distance in seconds of arc and θ the position angle in degrees as measured eastward from north.

As was usual for that day, before analysis could be performed, the equation of the apparent ellipse was transformed to a pair (x, y) of coordinates where x was north at 0° and y east at 90°. Thus

$$x = \rho \cos\theta, \qquad y = \rho \sin\theta. \tag{1.15}$$

In Cartesian coordinates, the general equation for an ellipse can be written as

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + 1 = 0. \tag{1.16}$$

From the observed data one can then perform a least-squares analysis to find the coefficients. Smart suggests the way to derive a solution graphically, but a modern computational approach is more effective.[4] Once the values of these coefficients are obtained, the orbital elements can be derived in a nontrivial way:

1. The nodal angle Ω is obtained from

$$\left(F^2 - G^2 + A - B\right) \sin 2\Omega + 2\,(FG - H) \cos 2\Omega = 0. \tag{1.17}$$

2. The inclination angle i and the semi-latus rectum $p = a\left(1 - e^2\right)$ are found by solving two equations,

$$FG - H = -\frac{\sin 2\Omega \tan^2 i}{2p^2}, \qquad F^2 + G^2 - (A + B) - \frac{\tan^2 i}{p^2} = \frac{2}{p^2}. \tag{1.18}$$

3. The argument of periapsis ω is found from

$$\tan\omega = \frac{(F\cos\Omega - G\sin\Omega)\cos i}{F\sin\Omega + G\cos\Omega}. \tag{1.19}$$

4. The orbital eccentricity e is found from

$$e = \frac{(G\sin\Omega - F\cos\Omega)\, p\cos i}{\sin\omega}. \tag{1.20}$$

5. The semi-major axis a is found from e and p,

$$a = \frac{p}{1 - e^2}. \tag{1.21}$$

6. The true anomaly ν is found for any point where θ is available using

$$\tan(\nu + \omega) = \tan(\theta - \Omega)\sec i. \tag{1.22}$$

7. The eccentric anomaly E is found from

$$\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}. \tag{1.23}$$

8. With the mean motion given by $n = 2\pi/T$, where T is the orbital period, the time difference from periastron $(t - \tau)$ for each observation is obtained from Kepler's equation,

$$n\,(t - \tau) = E - e\sin E. \tag{1.24}$$

[4] See **6-0VisBin** for examples of such computational solutions.

Obtaining the period T is a central problem in astronomy and astrophysics. Past observations were rarely equally spaced in time; therefore sophisticated Fourier methods that are so effective today could not be applied directly. Long periods are notoriously hard to determine, particularly with unequal interval methods.

A more modern approach can be found in Green (1985). Green derives the orbit parameters using the more sophisticated Thiele–Innes method. The Thiele–Innes method starts with the projection properties of the physical orbit upon the sky, and although its derivations are not given here, its computational structure can be somewhat more compact and is generally preferred in modern times.

It is clear that the process of obtaining individual masses of binary stars is an arduous one from visual data alone without spectroscopic, eclipsing, and interferometric observations to help with resolving the various ambiguities of the solutions. We address this topic in more detail in Chapter 6 when we consider the stellar motions in the N-body problem and in Chapter 8 where we consider the motions of stars around the galactic center.[5]

1.2.2 Spectroscopic binaries

If the radial velocity of one of the visual binary components can be determined from spectroscopic observations, then the angular orbital properties of i and Ω can be resolved directly. If radial velocity information is available for both stars, the mass ratio can also be obtained as an alternative to using astrometric data on proper motions. The radial velocities are not sufficient on their own to determine the masses, because the inclination is not derivable unless the system is also either a visual or eclipsing binary.

It is worth noting that spectroscopic methods can obtain the orbital period T, the eccentricity e, the daily motion n, and the argument of periapsis ω uniquely. They can also determine which nodes are which and the sign of the inclination when used in conjunction with visual or interferometric observations. The equation for orbital radial velocity (where z is along the line of sight, and $n = 2\pi/T_{days}$ is the daily motion) is

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{na\sin i}{\sqrt{1-e^2}}\left(\cos(\nu+\omega) + e\cos\omega\right). \tag{1.25}$$

The observed radial velocities will have the motion of the center of mass in each, and this is determined so that the line $v =$ constant divides the radial velocity curve into two equal areas. This has to be done so that $\mathrm{d}z/\mathrm{d}t$ is isolated.[6]

1.2.3 Other methods

Other examples of mass determination can be seen in a two-line binary (when neither star can be resolved, but a spectroscopic line can be obtained for both stars) or an eclipsing binary. However, detailed considerations of the spectroscopic methods used when there are two lines present or when a light curve is available for the star is beyond the scope of this book. In the case of two lines, the methods follow the one-line analysis in principle. Eclipsing binaries require an even more extensive elaboration

[5] Computational examples of visual binary orbits are given in **6-0VisBin** and in five notebooks in Chapter 8.
[6] Details can be found in **1-2SpectBin**.

and are not discussed here. See Smart (1960) or Green (1985) for descriptions. Green analyzes a binary pulsar as an interesting modern example and we consider this in Chapters 4 and 5.[7]

1.3 Kinetic theory

Many astrophysical models rely heavily on the behavior of fluid gases and plasmas. Although simple models often assume these to be an ideal gas, more sophisticated models require an examination of behavior at the particle level. Therefore an understanding of kinetic theory is central to many of these models. In this text we assume readers have at least a general understanding of thermodynamics and kinetic theory. For readers who have not taken a formal course in statistical thermodynamics or classical thermodynamics, we recommend going through the Wolfram *Mathematica*® notebooks on the subject in the appendix. They present the concepts of kinetic theory used in the text as well as demonstrating some basics of *Mathematica* programming.

1.3.1 Maxwell–Boltzmann statistics

To keep things as simple as possible, we consider a force-free monatomic dilute ideal gas. For a large collection of these "ideal" particles at a temperature T, the average kinetic energy of a particle is

$$\frac{1}{2}mv^2 = \frac{3}{2}kT, \tag{1.26}$$

where k is Boltzmann's constant. Although this relation defines temperature in terms of particle kinetic energy, speed v in this equation is an average speed of the particles. The collection of particles is distributed over many speeds, spread out about the average speed. The probability distribution for the particles is given by the Maxwell–Boltzmann distribution,

$$f(v) = 4\pi v^2 \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}, \tag{1.27}$$

such that the probability of finding a particle with a speed between v and $v + \mathrm{d}v$ is

$$\mathrm{d}p(v) = f(v)\,\mathrm{d}v. \tag{1.28}$$

The function is normalized so that

$$\int_0^\infty f(v)\,\mathrm{d}v = 1. \tag{1.29}$$

Random walk studies have shown that the Maxwell–Boltzmann equation is the equilibrium velocity distribution for dilute classical gases. McLennan (1989, p. 39) obtains the equation as a solution of the Fokker–Planck equation, whereas Mohling (1982) derives it from binary collision theory.[8]

If the system is not in thermodynamic equilibrium, changes in the velocity distribution due to external forces are described by the Boltzmann Transport Equation (BTE). The BTE allows in principle a complete specification of the transport equations required in most astrophysical situations although

[7] See, for example, **4-8ModelNS** and **5-6binarypulsar**.
[8] Computational examples of the Maxwell–Boltzmann distribution can be found in **7-5Maxwell**. Examples of the Boltzmann Transport Equation can be found in **7-6Boltzmann** and **7-7Collisions**.

there are hydrodynamic equations that will serve as well. Suitable expressions are derived in Reif (1965), Mohling (1982), and McLennan (1989). The standard form of the BTE is

$$\left[\frac{\partial}{\partial t} + \bar{v} \cdot \nabla_r + F_{ext} \cdot \nabla_p\right] f(\bar{r}, \bar{p}, t) = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}, \tag{1.30}$$

where $f(r, p, t)$ is the distribution function in position-momentum phase space. If the right-hand side of the equation vanishes, the system is said to be collisionless.

1.3.2 The partition function and Saha equation

The Maxwell–Boltzmann and related equations derive from the assumption that our gas particles are classical with no internal structure. However, at the quantum level the particles of a system can have discrete rather than continuous energy states. If one takes j as the index representing the possible discrete quantum states of a system, and E_j as the energy of the system in that state, then one may define the partition function for the system,

$$Z = \sum_j e^{-E_j/kT}, \tag{1.31}$$

which gives a distribution of particles in the quantum states. If there are multiple states that share the same energy E_j, then the system is said to be degenerate, and the partition function becomes

$$Z = \sum_j g_j e^{-E_j/kT}, \tag{1.32}$$

where g_j is known as the degeneracy factor. Partition functions are central to the Boltzmann–Gibbs–Helmholtz approach to thermal physics and thermodynamics. Essentially, it is assumed that one knows the quantum mechanical energy level structure for the most elementary component (say an "atom") of the system. Construction of the partition function for these "atoms" then leads to the macroscopic properties of ensembles consisting of those atoms.[9]

For a gas at high temperatures, thermal collisions can ionize a certain fraction of the atoms within the gas. Ionization equilibrium was a concept prevalent in astrophysics in the early part of the 20th century. It was given a quantitative status by the astrophysicist Saha through a derivation that is closely related to the law of mass action.[10] In thermal equilibrium, the excitation within the bound states is described by the Boltzmann distribution. The ionization is described by the Saha equation,

$$\frac{N_{y+1} N_e}{N_y} = \frac{Z_{y+1} Z_e}{Z_y} e^{-\chi/kT}, \tag{1.33}$$

where N_y represents the number density in the yth ionization (with y electrons removed), N_e is the electron density, the Zs are the respective partition functions, and χ is the ionization potential. The ionization potential corresponds to the molecular dissociation energies in the regular law of mass action. Although Saha's equation is fairly simple, it actually masks how complicated the pooled ionization from multiple atoms is to treat in practice.[11]

[9] Several examples of partition functions can be found in **6Partitionfs**, in the thermonotebooks directory of the appendix.
[10] For a discussion of the law of mass action, see **11MassAction**.
[11] See **12Ionization** for a more detailed discussion.

1.3.3 Fermi–Dirac statistics

At room temperature and higher the ideal gas law is usually a good description of a gas. However, once a gas is cooled beyond a certain point the classical rules no longer hold, even approximately. At low temperatures, known as the quantum degeneracy region, it is the particle spin forces that are dominant.

All simple (elementary) particles possess a quality known as spin. It has the same basic properties as angular momentum, except that it is quantized into discrete values. Particles that have even or zero spin states are called bosons. Particles that have total spins in multiples of 1/2 are called fermions. The most important manifestation of these spin states is its effect on the thermal average of particle occupancy obtained from the so-called Gibbs sum for non-dilute gases,

$$\langle N(\epsilon)\rangle = \frac{1}{\mathrm{e}^{(\epsilon-\mu)/kT} \pm 1}, \tag{1.34}$$

where ϵ is the energy of the state and μ is the chemical potential. The $+1$ form describes fermions, while the -1 form describes bosons. For ideal gases it is assumed that the exponential is much larger than 1; thus

$$\langle N(\epsilon)\rangle = \mathrm{e}^{-(\epsilon-\mu)/kT}, \tag{1.35}$$

which is the Maxwell–Boltzmann case.

Fermions by the Pauli exclusion principle can have occupancy states of 1 or 0, with the average occupancy bound by that range. At absolute zero, all energy states with energies less than the Fermi energy ϵ_F will be filled, and all states above the Fermi energy will be empty.

A simple calculation of the Fermi energy can be found by assuming electrons that are in a rigid cubic box (infinite potential well) of side a.[12] The energy states for such a box can be indexed by $\mathbf{n} = (n_x, n_y, n_z)$, and the energy states are then

$$E_{\mathbf{n}} = \frac{\hbar^2\pi^2}{2ma^2}\mathbf{n}^2. \tag{1.36}$$

The number of states with $E_{\mathbf{n}} < E_F$ are those that lie within a spherical volume of n_F; thus

$$N = 2\left(\frac{1}{8}\right)\frac{4\pi}{3}n_F^3. \tag{1.37}$$

The factor 2 is due to the two allowed spin states, while the 1/8 factor accounts for our need for only positive energy levels. From this one can express n_F in terms of the total number of electrons. From these we find

$$E_f = \frac{\hbar^2\pi^2}{2ma^2}\mathbf{n_F}^2 = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}, \tag{1.38}$$

where $n = N/a^3$ is the number density of electrons. The total energy of the system is then

$$E = \int E_F \mathrm{d}n = \frac{3}{5}NE_F, \tag{1.39}$$

and the average energy of the electrons is

$$\bar{E} = \frac{3}{5}E_F. \tag{1.40}$$

[12] Surprisingly, this is a reasonable approximation for electrons in a metal.

From the Fermi energy one can define a Fermi temperature,

$$T_F = \frac{E_F}{k}. \tag{1.41}$$

In most instances of a many-particle system, the calculated Fermi temperature in Kelvin is many orders of magnitude higher than the actual prevailing temperature. When that condition holds, a good approximation of the properties of fermions is simply to set $T = 0$. A better approximation in the vicinity of $T = 0$ is to use series approximations of the chemical potential as shown by Laurendeau (2010). More complex solutions can be obtained computationally.[13]

1.3.4 Bose–Einstein statistics

Bosons are not limited in their occupancy states. For bosons the energy distribution function (similar to the partition function) is

$$B(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/kT} - 1}. \tag{1.42}$$

The density of states for bosons of zero spin is

$$n(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)\sqrt{\epsilon}. \tag{1.43}$$

From this one can calculate the energy of the boson system,

$$E = \int_0^\infty B(\epsilon)n(\epsilon)\epsilon \, d\epsilon, \tag{1.44}$$

which yields

$$E = \frac{3V}{4\sqrt{2}}\left(\frac{m}{\pi\hbar^2}\right)^{3/2}(kT)^{5/2}\mathrm{Li}_{5/2}(e^{\mu/kT}), \tag{1.45}$$

where

$$\mathrm{Li}_s(x) = \sum_{i=1}^{\infty}\frac{x^i}{i^s} \tag{1.46}$$

is known as a polylog function. Although this is a complicated function it can be handled computationally fairly easily. *Mathematica*, for example, includes the function as $\mathrm{Li}_s(x) =$ `PolyLog[s,x]`.[14]

As the temperature of a boson gas approaches absolute zero, the system becomes degenerate. A degenerate boson gas collapses into the single lowest energy state. In a very real sense, they behave as if they are a single boson in the ground state, known as a Bose–Einstein condensate.

1.3.5 Black-body radiation

Photons are massless bosons and therefore follow Bose–Einstein statistics. The energy distribution for photons is usually expressed in terms of frequency or wavelength rather than energy; hence

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1}, \tag{1.47}$$

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}, \tag{1.48}$$

[13] See **13FermionsBosons**.
[14] For more detailed examples see **13FermionsBosons**.