Chapter 1

LOGICS WITH ACTUALIST QUANTIFIERS

In the possible-worlds models for quantified modal logic introduced by Kripke [1963b], each world w is assigned a set Dw, thought of as the domain of individuals that exist, or are *actual*, in w. Such models are said to have *varying domains*. A universal quantifier $\forall x$ is interpreted at w by taking the variable x to range over the domain Dw. This is called the *actualist* interpretation of quantification. It does not validate the *Universal Instantiation* scheme[4]

$$\forall x\varphi \to \varphi(y/x), \quad \text{where } y \text{ is free for } x \text{ in } \varphi,$$

because the value of variable y may not exist in a particular world. Kripke proposed instead to use the scheme

$$\forall y(\forall x\varphi \to \varphi(y/x)),$$

which is valid under the actualist interpretation, and which we will call *Actual Instantiation*. This chapter explores logics that have this axiom, and develops a new kind of "admissible" semantics for them.

The first two sections present the syntax of languages and logics. Section 3 is motivational, reviewing the way that admissible semantics has been developed to overcome incompleteness for propositional modal logics, and explaining the ideas behind our adaptation of this approach to logics with quantifiers. Section 4 reviews some of the historical background to quantifier notation and its algebraic interpretation. The next few sections set out the formal semantics and prove soundness and completeness theorems. The last section gives some examples of incompleteness, designed to show that the use of admissibility is unavoidable.

1.1. Syntax

To begin with, some notation will be established for the syntax of modal predicate logic with quantification of individual variables. We take as fixed

[4]We use "scheme" generically to mean a set of formulas that comprises all instances of a particular syntactic form.

a denumerable set $\mathsf{InVar} = \{v_0, \dots, v_n, \dots\}$ of such variables. Symbols like $x, y, z, x_1, x_1', \dots$ will often be used for arbitrary members of InVar.

A *signature* is a set $\mathcal{L}$ of individual constants c, predicate symbols P, and function symbols F. Each predicate or function symbol comes with an assigned positive integer, its *arity*, so is *n-ary* for some n. We write $\mathsf{Con}_{\mathcal{L}}$, or just Con, for the set of individual constants in $\mathcal{L}$.

Now we define the *terms* of a signature, for which we typically use the symbol τ, possibly with subscripts or superscripts. An $\mathcal{L}$*-term* is any variable from InVar, any constant c from $\mathsf{Con}_{\mathcal{L}}$, or inductively any expression $F\tau_1 \cdots \tau_n$ where F is an n-ary function symbol from $\mathcal{L}$, and $\tau_1, \dots, \tau_n$ are $\mathcal{L}$-terms. A *closed* term is one that has no variables, and so contains only constants and (possibly) function letters.

An *atomic* $\mathcal{L}$*-formula* is any expression $P\tau_1 \cdots \tau_n$ where P is an n-ary predicate symbol from $\mathcal{L}$, and $\tau_1, \dots, \tau_n$ are $\mathcal{L}$-terms. The set of $\mathcal{L}$*-formulas* is generated from the atomic ones and a constant formula F (Falsum) in the usual way, using the connectives $\wedge$ (conjunction), $\neg$ (negation), the modality $\Box$ and universal quantifiers $\forall x$ for each $x \in \mathsf{InVar}$. Other connectives $\vee$ (disjunction), $\to$ (implication), $\leftrightarrow$ (biconditional), $\Diamond$ (dual to $\Box$), and the existential quantifiers $\exists x$ are introduced by standard definitions:[5]

$$\begin{aligned}
\varphi \vee \psi &= \neg(\neg\varphi \wedge \neg\psi).\\
\varphi \to \psi &= \neg(\varphi \wedge \neg\psi).\\
\varphi \leftrightarrow \psi &= (\varphi \to \psi) \wedge (\psi \to \varphi).\\
\Diamond\varphi &= \neg\Box\neg\varphi.\\
\exists x\varphi &= \neg\forall x\neg\varphi.
\end{aligned}$$

The constant T (Verum) is defined to be $\neg\mathsf{F}$. Formulas are denoted by the symbols $\varphi, \psi, \chi, \dots$.

The size of the set of $\mathcal{L}$-formulas depends on the size of the signature $\mathcal{L}$. Even when $\mathcal{L} = \emptyset$ there are countably infinitely many formulas that can be generated from F. On the other hand, if $\mathcal{L}$ contains uncountably many constants, and at least one predicate symbol, it will have uncountably many atomic formulas. In general, for a signature with at least one predicate symbol, the number of formulas is equal to the maximum of $\aleph_0$ and the size of $\mathcal{L}$.

We also need to consider *propositional* modal logics, whose language we take to be generated by an infinite set $\mathsf{PropVar}$ of *propositional* variables, for which we use symbols like $p, q, p_1, q_1', \dots$. Formulas are constructed from these and F by the connectives $\wedge$, $\neg$ and $\Box$. We may use the letters $A, B, \dots$ for such *propositional modal formulas*. An $\mathcal{L}$-formula φ of quantified modal logic will be called a *substitution-instance* of some propositional formula A if φ

[5] Technically, we could define F as some formula of the form $\varphi \wedge \neg\varphi$, but it is convenient to have it as a primitive.

can be obtained by uniform substitution of $\mathcal{L}$-formulas for the propositional variables of A. In particular, an $\mathcal{L}$-formula φ will be called a *Boolean tautology* if it is a substitution-instance of some propositional formula that is valid in the two-valued semantics of Boolean propositional logic.

It is common to assume that the set $\mathsf{PropVar}$ of propositional variables is *countably* infinite, but the theory also allows it to be of any uncountable size. We make use of that option in Section 1.10, where we assume that there are at least as many propositional variables as there are $\mathcal{L}$-formulas for some possibly uncountable signature $\mathcal{L}$.

Extensive use will be made in this book of the operation of substituting terms for free occurrences of variables. This needs to be handled carefully, and we take some trouble over its notation now. *Free* and *bound* occurrences of a variable in a formula are defined in the usual way, and a *sentence* is a formula with no free variables. The symbol $\varphi(\tau/x)$ will be used for the formula obtained by replacing every free occurrence of the variable x in φ by the term τ. This standard notation conveys the idea of τ *overwriting* free x in φ.

We also need to consider the operation of making *simultaneous* substitutions for a number of free variables at once. The notation

$$(\tau_0/v_0, \dots, \tau_n/v_n, \dots)$$

will be used for the substitution operator that uniformly substitutes the term τ_n for all free occurrences of the variable v_n, and does this simultaneously for all $n \geq 0$. This operator can be applied to any formula φ to give a formula

$$\varphi(\tau_0/v_0, \dots, \tau_n/v_n, \dots).$$

It can also be applied to a term τ, to form $\tau(\tau_0/v_0, \dots, \tau_n/v_n, \dots)$. The notation may be abbreviated to

$$(\tau_{n_0}/v_{n_0}, \dots, \tau_{n_p}/v_{n_p})$$

to indicate that the substitution alters only $v_{n_0}, \dots, v_{n_p}$, i.e. $\tau_n = v_n$ for all $n \notin \{n_0, \dots, n_p\}$. We will make particular use of single-substitution operators of the type (c/x) that replace all free occurrences of the variable x by the constant c, leaving all other variables unchanged.

Note that simultaneous substitution may produce a different result to sequential compositions of single substitutions. Thus if φ is the atomic formula Pxy, then $\varphi(y/x, z/y)$ is Pyz, whereas applying (y/x) to φ, and then (z/y) to the result, gives Pyy and then Pzz. So $\varphi(y/x, z/y) \neq \varphi(y/x)(z/y)$. Also, this shows that composition of substitutions need not commute, as $\varphi(z/y)(y/x) \neq \varphi(y/x, z/y)$ in this example. However, if the substituting terms are closed, then the order of substitution is immaterial. Thus if τ, τ' are closed, then

$$(\tau/x, \tau'/x') = (\tau/x)(\tau'/x') = (\tau'/x', \tau/x).$$

This also holds more generally if x' is not in τ and x is not in τ'.

As is customary, τ is said to be *freely substitutable for x in φ*, or more briefly *free for x in φ*, if no free occurrence of x in φ is within the scope of a quantifier $\forall y$ where y is any variable occurring in τ. If this condition holds, then no variable in τ becomes bound in $\varphi(\tau/x)$ within the occurrences of τ that replace free x in φ.

Another kind of operator we will use is substitution for constants, writing $\varphi(\tau/\mathsf{c})$ for the result of replacing *every* occurrence of the constant c in φ by the term τ. τ is *free for* c *in* φ if no occurrence of c in φ is within the scope of a quantifier $\forall y$ where y is any variable occurring in τ. We will also use simultaneous substitutions $\varphi(\tau_1/\mathsf{c}_1, \dots, \tau_n/\mathsf{c}_n)$ of this kind.

Further substitution technology will be introduced later in Section 4.2.

1.2. Logics

The axiom schemes we need are as follows:

K: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

AI: $\forall y(\forall x\varphi \to \varphi(y/x))$, where y is free for x in φ.

UD: $\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$

VQ: $\varphi \to \forall x\varphi$, where x is not free in φ.

K, which is named for Kripke, is the basic axiom for propositional modal logics. AI is the scheme of *Actual Instantiation*, named for the *actualist* interpretation of the quantifiers, in a given world, as ranging over the individuals that actually exist in that world. UD is *Universal Distribution*, and VQ is *Vacuous Quantification*.

Inference rules will be displayed in the form

$$\frac{\varphi_1, \dots, \varphi_n}{\varphi}$$

A set of formulas is *closed under* such a rule if it contains the *conclusion* φ whenever it contains all of the corresponding *premisses* $\varphi_1, \dots, \varphi_n$.

The inference rules we use are

MP:	$\dfrac{\varphi,\ \varphi \to \psi}{\psi}$		*Modus Ponens*
N:	$\dfrac{\varphi}{\Box\varphi}$		*Necessitation*
UG:	$\dfrac{\varphi}{\forall x\varphi}$		*Universal Generalisation*
TI:	$\dfrac{\varphi}{\varphi(\tau/x)}$,	if τ is free for x in φ.	*Term Instantiation*

GC: $\dfrac{\varphi(\mathsf{c}/x)}{\varphi}$, if c is not in φ. *Generalisation on Constants*

For a given signature $\mathcal{L}$, a *quantified modal logic*, or more briefly a *logic*, is defined to be any set L of $\mathcal{L}$-formulas that includes all Boolean tautologies and instances of the schemes K, AI, UD and VQ, and is closed under the rules MP, N, UG, TI and GC. A member φ of L is called an L-*theorem*, which we indicate by writing $\vdash_{\mathrm{L}} \varphi$. Sometimes we write this as $\mathrm{L} \vdash \varphi$. We may also just write $\vdash \varphi$ if the logic in question is understood. If we need to be specific about the background signature involved, we may say that L is a logic *over* $\mathcal{L}$.

In working with a logic we may write "by PC", for "by Propositional Calculus", meaning that a result follows by reasoning available in Boolean propositional logic.

It is worth being aware from the outset that our definition of logics does not include the scheme

CQ: $\forall x \forall y \varphi \to \forall y \forall x \varphi$

of *Commuting Quantifiers*. We will have a good deal to say later about the role of this scheme as a further axiom.

The Term Instantiation rule would be derivable, using UG, in any logic that included the Universal Instantiation scheme

UI: $\forall x \varphi \to \varphi(\tau/x)$, where τ is free for x in φ.

But we are not assuming that axiom, which relates to the possibilist interpretation of the quantifiers, and will not analyse its role for some time (see Section 2.4). Neither of our rules TI and GC have been commonly taken as primitive in quantified modal logics, but we will see that both are sound for the semantics of varying-domain models, and both are needed for the construction of a characteristic model for an arbitrary logic. They are in fact derivable in many standard logics that are axiomatised by specified modal axioms, as we show at the end of this section. A number of important basic properties of logics do not depend on these two rules, as we show now.

LEMMA 1.2.1. *In any quantified modal logic* L, *the following can be shown without using* TI *or* GC.

(1) L *is closed under the following* $\forall$-*rules and* $\exists$-*rules*:

$\forall$-*Monotonicity*: $\dfrac{\varphi \to \psi}{\forall x \varphi \to \forall x \psi}$

$\forall$-*Equivalence*: $\dfrac{\varphi \leftrightarrow \psi}{\forall x \varphi \leftrightarrow \forall x \psi}$

$\forall$-*Introduction*: $\dfrac{\varphi \to \psi}{\varphi \to \forall x \psi}$, *if x is not free in φ.*

$\exists$-*Monotonicity*: $\dfrac{\varphi \to \psi}{\exists x \varphi \to \exists x \psi}$

$\exists$-*Equivalence*: $\dfrac{\varphi \leftrightarrow \psi}{\exists x\varphi \leftrightarrow \exists x\psi}$

$\exists$-*Elimination*: $\dfrac{\varphi \to \psi}{\exists x\varphi \to \psi}$, *if x is not free in ψ.*

(2) $\vdash_{\mathrm{L}} \forall x(\varphi \wedge \psi) \leftrightarrow \forall x\varphi \wedge \forall x\psi$.

(3) Replacement of Provable Equivalents:
If $\vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$, then $\vdash_{\mathrm{L}} \chi[\varphi] \leftrightarrow \chi[\psi]$, where $\chi[\varphi]$ and $\chi[\psi]$ differ only in that $\chi[\varphi]$ has φ in some places that $\chi[\psi]$ has ψ.

(4) *If x, y are distinct variables with y not free in φ but freely substitutable for x in φ, then $\vdash_{\mathrm{L}} \forall x\varphi \leftrightarrow \forall y\varphi(y/x)$.*

(5) Relettering of Bound Variables:
If φ and ψ differ only in that φ has free x exactly where ψ has free y, then $\vdash_{\mathrm{L}} \forall x\varphi \leftrightarrow \forall y\psi$.

PROOF. (1)–(3) are standard, with the $\exists$-rules of (1) following by PC from the $\forall$-rules. For (4), from Actual Instantiation by Universal Distribution and PC we get

$$\vdash \forall y\forall x\varphi \to \forall y\varphi(y/x).$$

But $\vdash \forall x\varphi \to \forall y\forall x\varphi$ by Vacuous Quantification as y is not free in $\forall x\varphi$, so by PC this implies $\vdash \forall x\varphi \to \forall y\varphi(y/x)$. For the converse implication, the hypotheses on x and y ensure that x is free for y in $\varphi(y/x)$, and that $\varphi(y/x)(x/y) = \varphi$, so as another instance of AI we have

$$\vdash \forall x(\forall y\varphi(y/x) \to \varphi).$$

Using UD and VQ again, this leads by PC to $\vdash \forall y\varphi(y/x) \to \forall x\varphi$, and hence to (4).

For (5), the assumptions mean that ψ is $\varphi(y/x)$, and (5) follows directly from (4). ⊣

The last results of this Lemma allow us to take an occurrence of a formula $\forall x\varphi$ as a subformula of some formula ψ and replace it by $\forall y\varphi(y/x)$ with y not occurring in $\forall x\varphi$. The result is a new formula, provably equivalent to ψ, in which these bound occurrences of x have been replaced by y. Any formula ψ' obtained from ψ by a sequence of such reletterings of bound variables is a *bound alphabetic variant* of ψ. Another term for this is "congruence". More precisely, formulas ψ and ψ' will be called *congruent* if there is a bijection $x_i \mapsto x_i'$ between their sets of bound variables such that ψ and ψ' are identical except that ψ has bound ocurrences of x_i exactly where ψ' has bound ocurrences of x_i' and vice versa. Using the above scheme, and the principle of replacement of provable equivalents (Lemma 1.2.1(3)), it can be shown, as in [Kleene 1952, Lemma 15b], that congruent formulas are provably equivalent, i.e:

LEMMA 1.2.2. *If ψ and ψ' are congruent, then the formula $\psi \leftrightarrow \psi'$ is derivable in any quantified modal logic.* ⊣

The rules TI and GC are not needed to show that this is so. We can use the result to follow the common procedure, when confronted by a term τ that is not free for some variable z in ψ, of replacing ψ by a bound alphabetic variant ψ' with no variable of τ being bound in ψ'. Then τ is free for z in ψ'. We discuss this further in Section 4.3.

Next we consider some consequences of the rules TI and GC.

Lemma 1.2.3. *Any logic* L *is closed under the following rules.*

TI*: $\dfrac{\varphi}{\varphi(\tau_1/x_1,\dots,\tau_n/x_n)}$, *if each* τ_i *is free for* x_i *in* φ.

GC*: $\dfrac{\varphi(\mathsf{c}_1/x_1,\dots,\mathsf{c}_n/x_n)}{\varphi}$, *if the* c_i *are distinct and not in* φ.

Sub: $\dfrac{\varphi}{\varphi(\tau/\mathsf{c})}$, *if* τ *is free for* c *in* φ.

Sub*: $\dfrac{\varphi}{\varphi(\tau_1/\mathsf{c}_1,\dots,\tau_n/\mathsf{c}_n)}$, *if each* τ_i *is free for* c_i *in* φ.

∀GC: $\dfrac{\varphi\to\psi(\mathsf{c}/x)}{\varphi\to\forall x\psi}$, *if* c *is not in* φ *or* ψ.

Proof. For TI*, note that if none of the variables x_i occur in any of the terms τ_j (for instance if the τ_j's are closed), then the conclusion of the rule can be obtained from φ by performing the single substitutions $(\tau_1/x_1),\dots,(\tau_n/x_n)$ sequentially. In this case the conclusion of TI* is obtained from φ by n applications of TI. For the general case, we apply this observation by first choosing fresh variables $y_1,\dots,y_n$ that do not occur in any of $\tau_1,\dots,\tau_n$, and then applying the $2n$ single substitutions $(y_1/x_1),\dots,(y_n/x_n),(\tau_1/y_1),\dots,(\tau_n/y_n)$ sequentially to reach the same conclusion (this argument can be found in Church 1956, p. 84).

For GC*, note first that since the c_i are closed, the substitutions involved in forming the premiss can be performed simultaneously, or sequentially in any order. We prove the result by induction on n. The case $n = 1$ is just GC itself. Assuming inductively the result for n, then if $\mathsf{c}_1,\dots,\mathsf{c}_{n+1}$ are distinct and not in φ, it follows that c_{n+1} is not in $\varphi(\mathsf{c}_1/x_1,\dots,\mathsf{c}_n/x_n)$, so if $\vdash\varphi(\mathsf{c}_1/x_1,\dots,\mathsf{c}_{n+1}/x_{n+1})$ then $\vdash\varphi(\mathsf{c}_1/x_1,\dots,\mathsf{c}_n/x_n)$ by GC, hence $\vdash\varphi$ by induction hypothesis. So the result holds for $n+1$. Thus it holds for all n by induction.

Sub is just the case $n = 1$ of Sub*, and we derive the latter directly. Suppose that $\vdash\varphi$ and τ_i is free for c_i in φ for all $i\le n$. Take distinct new variables $x_1,\dots,x_n$ that do not occur in φ. Then $\varphi(x_1/\mathsf{c}_1)(\mathsf{c}_1/x_1)=\varphi$, as x_1 is not in φ, so $\vdash\varphi(x_1/\mathsf{c}_1)(\mathsf{c}_1/x_1)$. From this we get $\vdash\varphi(x_1/\mathsf{c}_1)$ by GC, since c_1 is not in $\varphi(x_1/\mathsf{c}_1)$. Repeating this, from $\vdash\varphi(x_1/\mathsf{c}_1)$ we get $\vdash\varphi(x_1/\mathsf{c}_1)(x_2/\mathsf{c}_2)$, since x_2 is not in $\varphi(x_1/\mathsf{c}_1)$ and c_2 is not in $\varphi(x_1/\mathsf{c}_1)(x_2/\mathsf{c}_2)$. Sequentially applying the substitutions (x_i/c_i) in this way we arrive at $\vdash\psi$, where $\psi = \varphi(x_1/\mathsf{c}_1)(x_2/\mathsf{c}_2)\cdots(x_n/\mathsf{c}_n)$. In fact $\psi = \varphi(x_1/\mathsf{c}_1, \ldots, x_n/\mathsf{c}_n)$, by the distinctness of the x_i's and the c_j's.

Now for each $i \leq n$, the term τ_i is free for x_i in ψ, since it is free for c_i in φ and ψ has x_i exactly in the places where φ has c_i. So by rule TI* we get $\vdash \psi(\tau_1/x_1, \ldots, \tau_n/x_n)$. But $\psi(\tau_1/x_1, \ldots, \tau_n/x_n)$ is

$$\varphi(x_1/\mathsf{c}_1, \ldots, x_n/\mathsf{c}_n)(\tau_1/x_1, \ldots, \tau_n/x_n),$$

which is $\varphi(\tau_1/\mathsf{c}_1, \ldots, \tau_n/\mathsf{c}_n)$, again as the x_i's do not occur in φ. This completes the proof that L is closed under Sub*.

For $\forall$GC, suppose $\vdash \varphi \to \psi(\mathsf{c}/x)$ with c not in φ or ψ. Take a fresh variable y that does not occur in φ or ψ. By Sub, $\vdash (\varphi \to \psi(\mathsf{c}/x))(y/\mathsf{c})$. But this last formula is equal to $\varphi \to (\psi(y/x))$, so from it we can derive $\varphi \to \forall y\psi(y/x)$ by $\forall$-Introduction, as y is not free in φ (Lemma 1.2.1(1)). Since by Lemma 1.2.1(4) we have $\vdash \forall y\psi(y/x) \to \forall x\psi$, by PC this yields $\vdash \varphi \to \forall x\psi$ as required. ⊣

A *propositional modal logic* is a set S of propositional modal formulas that includes all such formulas that are Boolean tautologies or instances of the scheme K, and is closed under Modus Ponens and Necessitation.[6] Often such an S is presented as the *smallest*, or *least*, propositional modal logic that includes some specific set S_{ax} of formulas - typically the set of all instances of some axiom scheme(s), like $\Box A \to A$. In other words, S is the intersection of all propositional modal logics that include S_{ax}. When $\mathrm{S}_{ax} = \emptyset$, then S is the smallest of all propositional modal logics, and is commonly known as K. In general, theoremhood in S can be characterised by the existence of a finite proof-sequence in the usual way:

THEOREM 1.2.4. *Let* S *be the smallest propositional modal logic that includes some specified set* S_{ax} *of propositional modal formulas. Then* $\vdash_{\mathrm{S}} A$ *iff there exists a finite sequence* $A_1, \ldots, A_n = A$ *of propositional modal formulas such that each* A_i *is either a Boolean tautology; or an instance of the scheme* K; *or a member of* S_{ax}; *or follows from earlier members of the sequence by one of the rules* MP *and* N.

PROOF. This proceeds in a standard fashion by defining S′ to be the set of all propositional modal formulas A for which there exists a sequence $A_1, \ldots, A_n = A$ as described in the statement of the Theorem, and then observing that

(i) S′ is a propositional modal logic;
(ii) S′ includes S_{ax}; and
(iii) S′ is included in any propositional modal logic that satisfies (ii).

Then (i) and (ii) imply that $\mathrm{S} \subseteq \mathrm{S}'$, while (iii) ensures that $\mathrm{S}' \subseteq \mathrm{S}$. ⊣

[6] Actually this defines what is commonly called a *normal* propositional modal logic. But we will not be discussing non-normal systems in this book.

We will be particularly interested in logics that are quantified extensions of propositional ones. If S is any set of propositional modal formulas, we use the name QS for the *smallest* quantified modal logic that contains every $\mathcal{L}$-formula that is a substitution-instance of a member of S. In other words, QS is the intersection of all such quantified logics. Theoremhood in QS can also be characterised by the existence of a finite proof-sequence of a suitable type.

Theorem 1.2.5. *Let* S *be any set of propositional modal formulas.*

(1) $\vdash_{\mathrm{QS}} \varphi$ *iff there exists a finite sequence* $\varphi_1, \dots, \varphi_n = \varphi$ *of* $\mathcal{L}$*-formulas such that each* φ_i *is either an instance of a tautology or one of the schemes* K, AI, UD *and* VQ; *or a substitution-instance of a member of* S; *or follows from earlier members of the sequence by one of the rules* MP, N, UG, TI *and* GC.

(2) *If* S *is the smallest propositional modal logic that includes some set* S_{ax} *of propositional modal formulas, then* $\mathrm{QS} = \mathrm{QS}_{ax}$.

Proof. (1) is proved in an analogous manner to Theorem 1.2.4.

For (2), since $\mathrm{S}_{ax} \subseteq \mathrm{S}$, it is immediate that $\mathrm{QS}_{ax} \subseteq \mathrm{QS}$. For the converse inclusion it is enough to show that QS_{ax} contains all $\mathcal{L}$-formulas that are substitution-instances of members of S, since QS is the least such logic to do so.

So let A be a member of S with propositional variables $p_1, \dots, p_n$, and φ an $\mathcal{L}$-formula obtained by uniform substitution of some $\mathcal{L}$-formula φ_i for p_i in A, for all $i \leq n$. Let $(\varphi_1/p_1, \dots, \varphi_n/p_n)$ denote the substitution operator that substitutes each φ_i for p_i, and replaces any other propositional variables by F. Thus $\varphi = A(\varphi_1/p_1, \dots, \varphi_n/p_n)$.

Now let $A_1, \dots, A_n = A$ be a finite proof-sequence establishing $\vdash_{\mathrm{S}} A$ as given by Theorem 1.2.4. Then

$$A_1(\varphi_1/p_1, \dots, \varphi_n/p_n), \dots, A_n(\varphi_1/p_1, \dots, \varphi_n/p_n) = \varphi$$

is a finite sequence of $\mathcal{L}$-formulas each of which is (i) a substitution-instance of either a tautology or an instances of the scheme K or a member of S_{ax}; or (ii) follows from earlier members of the sequence by one of the rules MP and N, as substitution operators maps instances of MP and N to instances of the same rules. By part (1) of this Theorem, it follows that $\mathrm{QS}_{ax} \vdash \varphi$ as required. $\dashv$

The smallest quantified modal logic is QK, where K is the smallest propositional modal logic, and theoremhood in QK is characterised by proof-sequences as per Theorem 1.2.5(1) in which S is the set of instances of the axiom scheme K.

The adjective "propositional" will usually be used when referring to propositional modal logics, while the term "logic" used by itself can be taken to mean a quantified modal logic, unless the context clearly indicates otherwise.

Use of the letters "S" and "L" should help avoid confusion about which kind of logical system we are discussing.

The notation $\mathrm{QS} + \Sigma$ will often be used for the smallest quantified modal logic extending QS that includes a scheme Σ, as in $\mathrm{QS} + \mathrm{CQ}$, $\mathrm{QS} + \mathrm{UI}$ etc.

The deductive machinery introduced by Kripke [1963b] for the logic of varying domain model-structures consists essentially of the axioms and rules introduced here, with the exception of Term Instantiation and Generalisation on Constants. But the systems considered in that paper are built on propositional logics S in a similar manner to QS, and are of a kind in which these two extra rules are derivable.

To put it another way, we could have defined QS from S without using the rules TI and GC, and then shown that these rules are derivable.[7] To see how this works, write $\mathrm{Q'S} \vdash \varphi$ to mean that there exists a finite sequence $\varphi_1, \dots, \varphi_n = \varphi$ of $\mathcal{L}$-formulas such that each φ_i is either an instance of a tautology or one of the schemes K, AI, UD and VQ; or a substitution-instance of a member of S; or follows from earlier members of the sequence by one of the rules MP, N, and UG only. Call such a sequence a Q′S-*proof sequence*. It is immediate that $\mathrm{Q'S} \vdash \varphi$ implies $\mathrm{QS} \vdash \varphi$. To show the converse is true, it suffices to prove that $\{\varphi : \mathrm{Q'S} \vdash \varphi\}$ is closed under TI and GC. We outline a proof of this, leaving its fine details to the interested reader.

For TI, suppose $\varphi_1, \dots, \varphi_n = \varphi$ is a Q′S-proof sequence showing that $\mathrm{Q'S} \vdash \varphi$, and that τ is free for x in φ. By systematically relettering bound variables, we can turn this into a sequence $\varphi'_1, \dots, \varphi'_n$ that is still a Q′S-proof sequence such that each φ'_i is congruent to φ_i and τ is free for x in φ'_i. Then $\varphi'_1(\tau/x), \dots, \varphi'_n(\tau/x)$ will also be a Q′S-proof sequence, showing that $\mathrm{Q'S} \vdash \varphi'_n(\tau/x)$. Now we observed earlier that congruent formulas can be proved equivalent without using TI or GC (see Lemma 1.2.2). Since φ'_n and φ are congruent and τ is free for x in both, it follows that $\varphi'_n(\tau/x)$ and $\varphi(\tau/x)$ are congruent, hence $\mathrm{Q'S} \vdash \varphi'_n(\tau/x) \leftrightarrow \varphi(\tau/x)$. Therefore $\mathrm{Q'S} \vdash \varphi(\tau/x)$, as required by the rule TI.

For GC, let $\varphi_1, \dots, \varphi_n = \varphi(\mathsf{c}/x)$ be a Q′S-proof sequence showing that $\mathrm{Q'S} \vdash \varphi(\mathsf{c}/x)$, where c is not in φ. Take a new variable y that does not occur in this sequence or in φ. Then $\varphi_1(y/\mathsf{c}), \dots, \varphi_n(y/\mathsf{c})$ will also be a Q′S-proof sequence, showing that $\mathrm{Q'S} \vdash \varphi_n(y/\mathsf{c})$. Since neither c nor y occur in φ, the formula $\varphi_n(y/\mathsf{c})$, i.e. $\varphi(\mathsf{c}/x)(y/\mathsf{c})$, is just $\varphi(y/x)$, so $\mathrm{Q'S} \vdash \varphi(y/x)$. But x is freely substitutable for y in $\varphi(y/x)$, so $\mathrm{Q'S} \vdash \varphi(y/x)(x/y)$ by the rule TI just derived. Since $\varphi(y/x)(x/y) = \varphi$, this gives $\mathrm{Q'S} \vdash \varphi$ as required by the rule GC.

[7]Technically, the machinery of [Kripke 1963b] required the axioms to be sentences (no free variables), and the set of axioms to be closed under prefixing of quantifiers $\forall x$ as well as $\Box$. Only Modus Ponens needed to be taken as an inference rule, since the rules N and UG are then derivable using the axioms K and UD respectively.