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**188 Modern Approaches to the Invariant-Subspace Problem**

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# Modern Approaches to the Invariant-Subspace Problem

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# Preface

There is an outstanding problem in operator theory, the so-called 'invariant-subspace problem', which has been open for more than half a century. There have been significant achievements on occasion, sometimes after an interval of more than a decade, but its solution seems nowhere in sight. The invariant-subspace problem for a complex Banach space $\mathcal{X}$ of dimension $> 1$ concerns whether every bounded linear operator $T : \mathcal{X} \to \mathcal{X}$ has a non-trivial closed $T$-invariant subspace (a closed linear subspace $\mathcal{M}$ of $\mathcal{X}$ which is different from both $\{0\}$ and $\mathcal{X}$ such that $T(\mathcal{M}) \subset \mathcal{M}$). Throughout this book, when we talk about invariant subspaces, we always assume that they are closed and non-trivial.

For the most important case of Hilbert spaces $\mathcal{H}$ the problem is still open, although Enflo [95, 96] and Read [168, 169] showed that the invariant-subspace problem is false for some Banach spaces.

The general case of the invariant-subspace problem is still open, but there are many positive results in this direction. For example, every finite-rank operator on a non-zero complex space has an eigenvector, and this generates a one-dimensional invariant subspace. Thus the conjecture is easily resolved in the case that the underlying Hilbert space is finite-dimensional. Moreover, every non-zero vector is contained in a smallest invariant subspace, the *cyclic subspace* it generates, which is separable. Thus the question is easily answered for non-separable Hilbert spaces.

Aronszajn and Smith [18] proved that every compact operator on a Banach space of dimension at least 2 has an invariant subspace; the result for Hilbert spaces had already been proved by von Neumann (unpublished work of 1950). It is also known that normal operators have invariant subspaces: this follows easily from the spectral theorem.

In 1966, Bernstein and Robinson [39] proved, using non-standard analysis, that if the operator $T$ on a Hilbert space is polynomially compact (in other words, if $p(T)$ is compact for some non-zero polynomial $p$) then $T$ has an invariant subspace. Almost immediately, Halmos [117] eliminated the non-standard analysis from their proof and thus obtained a classical proof of this result.

These results were generalized further when Lomonosov [145] gave a very short proof, using the Schauder fixed-point theorem, of a powerful result which implies that if the operator $T$ on a Banach space commutes with a non-zero compact operator then it has a non-trivial invariant subspace (clearly, any operator $T$ commutes with every polynomial in $T$).

On the other hand, there are now many negative results known in the case of an operator on a Banach space. An example of an operator on a Banach space with no invariant subspaces was found by Enflo [95, 96], and his example was simplified by Beauzamy [28]. Another counterexample was given by Read [168], who later gave the first example on a 'classical' Banach space, namely $\ell^1$ [169]. In further work, Read [171] constructed an operator on $\ell^1$ without even a non-trivial closed invariant *subset*. We mention also some work of Atzmon [19], who gave an example of an operator without invariant subspaces on a nuclear Fréchet space.

Another very recent example is due to Argyros and Haydon [17], who constructed an infinite-dimensional Banach space such that every bounded operator is the sum of a compact operator and a scalar operator (i.e, a multiple of the identity). Therefore, in particular, every operator on this space has an invariant subspace. Gowers and Maurey [114] had earlier found a space where every bounded operator is the sum of a strictly singular operator and a scalar operator, but since Read [172] had already given an example of a strictly singular operator without invariant subspaces, the Gowers–Maurey example had no direct consequences for the invariant-subspace problem.

Some classic texts which treat the invariant-subspace problem are the books of Radjavi and Rosenthal [167] and Beauzamy [29]. Many other significant results can be found in the books of Herrero [122, 16]. A recent book closely linked to the theme of this book is that of Bayart and Matheron [26].

This book is intended to be suitable for researchers in many areas of operator theory and function theory. It should be accessible to postgraduate students; indeed, we begin with a section establishing the basic background results on Hardy spaces, operator theory, geometry of Banach spaces etc. We omit the proofs, which can be found in many places, for example Rudin [180, 179], Conway [84] and Nikolski [154].

There have been many significant developments in this branch of operator theory. Therefore, it was necessary to be selective in our choice of material. Some themes to be discussed in the book, which are not, as far as we know, to be found elsewhere in books, include the following:

- Use of the operator-valued Poisson kernel, factorizations, $H^\infty$ functional calculus (initiated by S. Brown in a Hilbert space context, as the basis of dual-algebra theory). Very recent Banach space developments due to Ambrozie and Müller.
- Functional calculus for Beurling algebras, the Atzmon–Wermer results. A more transparent presentation of Davie's work on Bishop operators (and his own functional calculus), and subsequent developments.
- The work of Lomonosov and Simonič, applying fixed-point theorems to the study of compact and essentially self-adjoint operators.
- The technique of minimal vectors pioneered by Ansari and Enflo in a Hilbert space context; further developments in the general context of Banach spaces, showing the existence of non-trivial hyperinvariant subspaces in various contexts.
- Universal operators. The study of composition operators and their invariant subspaces as key examples.
- Moment sequences, and the geometrical approach of Atzmon and Godefroy, with striking applications to tridiagonal matrices. Binomial sums and their applications in various areas of functional analysis.
- Particular results applying to special kinds of operators, such as positive (order-preserving) and strictly singular operators.

Chapter 1 contains the basic background material for the book. After this, Chapters 2, 3 and 4 follow a logical sequence and should be read in this order. The other chapters are largely independent and self-contained, with occasional references to one another. For example, compact operators appear in Chapters 6 and 7. Some supplementary results are contained in the Exercises at the end of each chapter.

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*Finally, this book is dedicated to our parents.*