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Excerpt
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1 Introduction

1.1 Introduction to Hilbert spaces

1.1.1 The basic idea

Hilbert spaces are the means by which the "ordinary experience of Euclidean concepts can be extended meaningfully into the idealized constructions of more complex abstract mathematics" (Bernkopf, 2008).

If our global plan is to abstract Euclidean concepts to more general mathematical constructions, then we better think of what it is in Euclidean space that is so desirable in the first place. An answer is geometry — in geometry one talks about points, lines, distances and angles, and these are familiar objects that our brains are well-adept to recognize and easily manipulate. Through imagery we use pictures to visualize solutions to problems posed in geometry. We may still follow Descartes and use algebra to furnish a proof, but typically through spatial reasoning we either make the breakthrough or see the solution to a problem as being plausible. Contrary to any preconception you may have, Hilbert spaces are about making obtuse problems have obvious answers when viewed using geometrical concepts.

The elements of Euclidean geometry such as points, distance and angle between points are abstracted in Hilbert spaces so that we can treat sets of objects such as functions in the same manner as we do points (and vectors) in 3D space. Hilbert spaces encapsulate the powerful idea that in many regards abstract objects such as functions can be treated just like vectors.

To others, less fond of mathematics, Hilbert spaces also encapsulate the logical extension of real and complex analysis to a wider sphere of suffering.

The theory of Hilbert spaces is one that succeeds in drawing together apparently different theories under a common framework via abstraction. The value of studying Hilbert spaces is not in providing new tools, but in showing how simple and familiar tools can be employed to tackle broad classes of problems. As a theory it highlights that there is a huge amount of redundancy in the literature. As a theory it requires relatively few ideas, but those ideas are deep.

1.1.2 Application domains

The usual application domains for Hilbert spaces are integral and differential equations, generalized functions and partial differential equations, quantum mechanics, orthogonal polynomials and functions, optimization and approximation theory. In signal processing and engineering: wavelets, optimization problems,

Euclid of Alexandria, (c. 300 BC) — "Euclid" is the anglicized name of Eukleides, who lived around 300 BC and is the "Father of Geometry," a title bestowed because of his very influential "Elements," which has served as a textbook on geometry and mathematical reasoning for more than two millennia. The style of Elements is in the form of definitions, axioms, theorems, and proofs. To the mathematically challenged this must come as the most torturous manuscript imaginable.

Not much is known about Euclid and his life, but along with Archimedes he is regarded as one of the greatest ancient mathematicians. His approach to geometry, which was seen as capturing directly statements about physical reality, has led to the terminology "Euclidean space." Over 23 centuries Euclidean space has been re-visited, recast, refined and generalized with notable extensions being "analytic geometry" (Cartesian coordinates) of René Descartes (1596–1650), and an axiomatic system for geometry, the 1899 "Grundlagen der Geometrie," of David Hilbert. The generalization of Euclidean space in the field of functional analysis is associated with a different abstraction by Hilbert and other researchers in the 1900s and called Hilbert space.

optimal control, filtering and equalization, signal processing on 2-sphere, Shannon information theory, communication theory, linear and nonlinear stability theory, and many more.

1.1.3 Broadbrush structure

Notion (cocktail party definition). *A Hilbert space is a* complete inner product space. *This is fine, except we are yet to define precisely what we mean by space, inner product and the adjective complete. But at a cocktail party where the objective is to impress strangers, particularly of the opposite sex, then it doesn't matter.* □

Notion (broad definition). *The term "vector" is ingrained in early mathematical education as an ordered finite list of scalars, but in Hilbert space it is a more general notion. We will alternatively use the term point in lieu of vector when the situation is not ambiguous. So when working in Hilbert spaces the word vector might represent a conventional vector, a sequence or a function (and even more general objects).* □

There are four key parts to a Hilbert space: vector space, norm, inner product and completeness. We can hear the minimalists screaming already.[1] To have a degree of comfort with Hilbert space is to have a clear notion of what these four things really mean and we will shortly move in the direction to address any deficiency. For the moment we are only interested in knowing what these mean in a general, possibly vague, way.

Vector space

Vector spaces should be familiar and align with the notions developed when dealing with the arithmetic of conventional vectors. Given two N-dimensional complex-valued vectors $a = (\alpha_1, \alpha_2, \ldots, \alpha_N)'$ and $b = (\beta_1, \beta_2, \ldots, \beta_N)'$, where $'$ denotes transpose, vector spaces encapsulate the banal aspects like

$$\begin{aligned} \gamma a + \delta b &= \gamma(\alpha_1, \alpha_2, \ldots, \alpha_N)' + \delta(\beta_1, \beta_2, \ldots, \beta_N)' \\ &= (\gamma\alpha_1 + \delta\beta_1, \gamma\alpha_2 + \delta\beta_2, \ldots, \gamma\alpha_N + \delta\beta_N)', \quad \gamma, \delta \in \mathbb{C}, \end{aligned} \tag{1.1}$$

where $\mathbb{C}$ denotes the set of complex numbers. (Also, $\mathbb{R}$ denotes the set of real numbers.)

Norm

The norm is a means to measure the size of vectors and define "convergence" when we have sequences of vectors. The norm generalizes the notion of Euclidean distance in $\mathbb{R}^N$

$$\|a\| = \left(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_N^2\right)^{1/2}. \tag{1.2}$$

[1] One of the drivers in mathematics is to provide a minimalist list of notions in preference to something that might be clearer and less clever. Engineers tend to think in terms of robustness and redundancy and this is our preferred approach in this book.

When armed with a norm we have a means to determine the distance between two vectors. The norm is responsible for a substantial part of the action in Hilbert spaces. It provides a measure of closeness, defines convergence and is necessary to make sense of "completeness."

Inner product

Inner product is a means to abstractly define orthogonality of vectors, projections and angles. We point out now that an inner product will be defined in such a way that it naturally induces a norm. That is, in the above list we are not implying that it is necessary to specify a norm in addition to specifying the inner product.[2] In Euclidean space $\mathbb{R}^N$, an inner product can be defined as

$$\langle a, b \rangle = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \cdots + \alpha_N \beta_N, \tag{1.3}$$

which induces (1.2) through $\|a\| = \langle a, a \rangle^{1/2}$. When armed with an inner product we can do everything we can with a norm. Finally, we mention that the inner product is also called a scalar product or the dot product.

Completeness

Completeness is a subtle concept associated with the norm to guarantee the vector space is big enough by including the natural limits of converging vector sequences.

Remark 1.1 (Banach space). In the above list were we to discard the inner product we still end up with something quite powerful and useful, known as a *Banach space*, named after Stefan Banach (1892–1945). A Banach space is a *complete normed space*. This means a vector space equipped with a norm and we have completeness. All Hilbert spaces are Banach spaces, but not all Banach spaces are Hilbert spaces. In a Banach space the norm needs to satisfy an additional condition known as the parallelogram equality to be a Hilbert space. That is, there is a condition on the norm that has a geometric interpretation. Loosely, we would say the norm needs to be a Euclidean-like norm and emulate Euclidean distance, but generally in a more abstract setting. When any result in Hilbert space does not strictly require the existence of an inner product, then that result will naturally belong to Banach space theory. □

Remark 1.2 (Finite-dimensional spaces). There exists a rich set of texts and works which make the theory of finite-dimensional vector spaces painfully bland. Hilbert spaces subsume such finite-dimensional vector spaces.

Be wary of salesmen trying to sell you completeness in a finite-dimensional space — completeness is automatic if the Hilbert space is finite-dimensional. More generally any finite-dimensional Banach space is automatically complete.

[2] So in the Hilbert space context the norm is like a free set of steak knives, it comes at no cost.

This is actually a non-trivial result. In real analysis, the property that a sequence of real numbers which is Cauchy necessarily converges is the nub of completeness. In summary, real numbers can be shown to be complete and finite-dimensional spaces inherit that completeness. □

Function spaces

The arena where the Hilbert space concept brings new insights is in the treatment of functions, that is, where our "points" are now whole functions in a *function space*. This means we are generally (but not always) considering *infinite-dimensional vector spaces*. Hilbert spaces do not need to be infinite-dimensional, but represent a degree of overkill in abstraction when we want to consider finite-dimensional vector spaces.

The preferred way to think about functions in Hilbert space is as points or vectors in space rather than as a mapping. So in functional analysis, when one says a "point or vector in Hilbert space" one means a "function." The mental image is a mathematical point in space (albeit infinite-dimensional space) — akin to a conventional vector — not a squiggly line. Using the conventional vector as an analog to guide thinking is very effective. That is, when dealing with Hilbert space, it is very profitable to repeatedly ask the question: what does this correspond to in the case of a finite-dimensional vector space?

Remark 1.3. A finite N-vector can be regarded as a function defined on $\{1, 2, \ldots, N\}$. Therefore, the terminology function space and functions in Hilbert space can be used to cover both cases of finite-dimensional and infinite-dimensional spaces. □

It is tempting to think that function spaces must be infinite-dimensional or that infinite-dimensional Hilbert spaces must be function spaces. Both associations are wrong as we now illustrate.

First, one example of a Hilbert space has elements that are sequences of real numbers which satisfy certain conditions

$$\{\alpha_1, \alpha_2, \alpha_3, \ldots\},$$

which can be added and scaled in obvious ways. When performing algebraic manipulations on sequences, they should be treated as column vectors of infinite size. This Hilbert space is infinite-dimensional, but does not involve what is generally understood to be functions.

The second space we shall consider, to sharpen our thinking, is the space of linear combinations of two functions, $1/2\pi$ (constant function) and $(1/\pi)\cos\theta$. That is, all elements of the space look like

$$f(\theta) = \frac{\alpha}{2\pi} + \frac{\beta}{\pi}\cos\theta, \quad \alpha, \beta \in \mathbb{R}.$$

In the end, computations regarding the norms and inner products of such elements reduce to linear algebra of 2-vectors (α, β) and as such are not different from $\mathbb{R}^2$. So even though this is a function space, it is only two-dimensional.

Problem

1.1. What besides a conventional vector, a sequence or a function might be further examples for an abstract vector?

1.1.4 Historical comments

The theory of Hilbert spaces was initiated by David Hilbert (1862–1943), in the early twentieth century in the context of the study of *integral equations*.[3] Of course, he did not decide to write a few papers and name the theoretical construct after him. Nor did he solely develop the theory. It is generally regarded that a number of people developed the theory of Hilbert spaces, especially Erik Ivar Fredholm (1866–1927), whose work directly influenced Hilbert. Hilbert's work was simplified, generalized and abstracted further by Hilbert's student Erhard Schmidt (1876–1959). Other researchers at the same time developed key results including Frigyes Riesz (1880–1956) in his work (Riesz, 1907) and Ernst Sigismund Fischer (1875–1954) in his contribution (Fischer, 1907). The term "Hilbert space," or at least the German equivalent, is generally attributed to John von Neumann (1903–1957) in 1929.

It is tempting to regard Hilbert space theory as a generalization of familiar Euclidean space, which is true. Yet the theory developed quite late in mathematics. The delays in the development of the theory were manyfold and we will step through these as they align with conceptual barriers that need to be hurdled to fully understand the theory at a sufficiently advanced level.

The theory came together due to a number of factors. The first factor related to how to deal with the *infinite* (which will be explored more fully below). The second factor concerned the ongoing dispute about the meaning of *Fourier series*, developed by Jean Baptiste Joseph Fourier (1768–1830). That is, what class of functions does the Fourier series expansion converge to and how does this relate to the original function. The third factor concerned *integration*. For function spaces the inner product is defined in terms of integrals. However, a sufficiently general notion of an integral was late in arriving and the preferred notion was due to Henri Lebesgue (1875–1941) in the early twentieth century.[4] The Lebesgue integral (and associated measure theory) is needed for a rigorous development of the most useful classes of Hilbert spaces. However, the more subtle aspects of the Lebesgue integral are not essential to come to grips with Hilbert space from an application viewpoint. We will, however, highlight the nature of the subtleties later. Finally, we mention that the development of Hilbert spaces received a significant boost from the co-development of quantum mechanics in the 1920s, largely through the work on operators by von Neumann.[5]

[3] Integral equations are a natural complement to differential equations and arise, for example, in the study of existence and uniqueness of functions which are solutions of partial differential equations such as the wave equation. Convolution and Fourier transform equations also belong to this class.

[4] Lebesgue is pronounced in the French style, that is, with pursed lips, with every second letter silent and the remainder mumbled.

[5] Again the most useful type of operator, called a compact operator, emerged much earlier and, in fact, inspired Hilbert to develop his first results in Hilbert spaces.

Hilbert, David (1862–1943) — Hilbert was a famous German mathematician who originally hailed from Königsberg (no longer part of Germany and now renamed) and spent the majority of his life in Göttingen. His biography records that he had a large 6m blackboard constructed in his back-yard along with a covered walkway, which enabled him to do his work outdoors in any weather. At the age of 45, he learned to ride a bike and combined riding with weeding and pruning trees as part of a ritualistic style of behavior when deep in thought on some mathematical problem in his back-yard (Reid, 1986). He had a propensity to get elementary work, such as calculus, garbled and confused in lectures sometimes leading to fiascos. He was described as "slow to understand" (Reid, 1986, p. 172). His thinking was more strongly directed towards existence-style arguments versus constructive ones. He elucidated the difference in lectures by saying "Among those who are in this lecture hall, there is one who has the least number of hairs."

Hilbert was deeply influential in the development of many fields of mathematics and mathematical physics. His influence came through the problems he worked on, his major breakthroughs, and making Göttingen a major center for mathematical research. The disproportionate strength of German mathematics in the world scene at the time is a tribute to Hilbert and Felix Klein (1849–1925) who shared the vision of establishing Göttingen as the world's leading mathematics research center. The importance of such individuals was driven home by the virtual destruction of Göttingen with political and social changes, which caused many key people, who were to carry on Hilbert's legacy, to leave Germany. It was a tragedy for Hilbert to see in 1934 (after his retirement) what he, Klein and Hermann Minkowski (1864–1909) had built was destroyed in a short period when he lamented (possibly in anger) "Mathematics in Göttingen? There is really none any more."

1.2 Infinite dimensions

It has been eluded to that the more relevant and interesting Hilbert spaces are infinite-dimensional. In Hilbert space theory it is critical to have the correct notion of infinity and it is not sufficient to regard the symbol ∞ as being something obvious. In the following, we are going to explore the meaning behind the various types of infinity. Although seemingly a large digression, it is only conceptually challenging and not technically challenging. It is highly relevant to understanding infinite-dimensional Hilbert spaces.

Sanity is optional

Mathematicians had a lot of trouble dealing with the infinite sets and finding the most sensible approach to the topic was left to Georg Cantor (1845–1918) in the late nineteenth century, who explored the boundary between sanity and insanity (Dauben, 1990; Aczel, 2001). To understand his *transfinite cardinals*[6] does not cause insanity, but it probably helps to be well down that track. Cantor's ideas met initially a lot of resistance, but now are seen to be profound (or at least profoundly crazy). Hilbert was greatly influenced by the ideas of Cantor, as might be gleaned from the problem of the "Continuum Hypothesis," which was first amongst Hilbert's list of 23 unsolved problems in the Paris conference of the International Congress of Mathematicians in 1900.[7]

1.2.1 Why understand and study infinity?

We now consider some attributes about infinite sets which underpin the structure of infinite-dimensional Hilbert spaces. In case you have met the theory of transfinite cardinals before and you want to skip this material then the coverage is: countable or denumerably infinite (transfinite) sets, $\aleph_0$; cardinality of the continuum; integers are countable but not dense in the reals; rationals are countable *and* dense in the reals; the continuum has a cardinality which is equivalent to the set of all subsets of a countable set, which may be written $\mathfrak{c}$; the existence of transfinite cardinals beyond $\mathfrak{c}$; etc. That is, to skip this material is only recommended if you have a familiarity with the general conceptual and arithmetical properties of transfinite cardinals.

Of critical importance in what is known as a *separable Hilbert space* is the existence of a *countable dense set*. Having a *countable* set of vectors and having

[6] Transfinite means beyond finite, i.e., infinite. The expression transfinite is preferred over infinite since it is less well recognized and, therefore, more likely to impress strangers. In short, the theory says there are different sizes of infinity with the smallest corresponding to the cardinality of the integers.

[7] Initially Hilbert had ten problems which were later expanded to 23. The Continuum Hypothesis is the hypothesis that there is no infinite set whose cardinality or size is strictly between that of the integers and that of the real numbers. The meaning behind the Continuum Hypothesis can be easily understood in the context of these notes. Kurt Gödel (1906–1978) showed in 1940 that the Continuum Hypothesis cannot be disproved from the standard set theory axiom system. Paul Joseph Cohen (1934–2007) showed in 1963 that the Continuum Hypothesis cannot be proven from those same axioms either. Therefore, mathematicians regard the problem of the Continuum Hypothesis resolved. Gödel starved himself to death believing people were trying to poison him.

Cantor, Georg (1845–1918) — Cantor was a German mathematician and born in Saint Petersburg, Russia. He initiated the theory of sets and the theory of transfinite numbers/cardinals. Famously, Cantor was institutionalized to an asylum a number of times most likely suffering depression brought on by being unremittingly assailed by his contemporaries in mathematics. His correspondence with colleagues reflects that he and his work were under constant criticism, particularly from Leopold Kronecker (1823–1891) in Berlin.

Cantor was involved in generating innovative and philosophically deep work, which challenged conventional thinking. Hilbert was one of his supporters, recognizing the significance of the work, and remarked, albeit somewhat too late in 1925 when giving a talk on “On the infinite” for a celebration in honor of Karl Weierstrass (1815–1897), “No one shall drive us out of this paradise that Cantor created for us” (Reid, 1986, pp. 176–177). It is now generally regarded that his work was a building block of modern mathematics and revolutionized many mathematical fields. Had he known what his impact was, then he might have enjoyed a better fate than dying in an institution in 1918. A comprehensive and scholarly biography of Cantor and his work is (Dauben, 1990) and a more popular account can be found in (Aczel, 2001).

that set *dense* yields a type of spanning property giving a natural generalization of what happens in finite-dimensional spaces. Knowing what *countable* and *dense* mean is important and will be explained later. These concepts derive from and are mimicked in the simpler analogous structure of rational numbers within the real numbers. The analogy is so faithful and can guide our intuition in function spaces and hence it justifies a digression to hone the concepts.

1.2.2 Primer in transfinite cardinals

Primary or elementary school children know infinity is pretty big. Some know that infinity plus one, two times infinity and infinity squared are at least as big or even bigger.[8] But some grown-ups have doubts. What is the nature of any formal assertion involving infinity? Ultimately how do you measure infinite sets or compare infinite sets? Satisfactory resolution of these questions had to wait for the Theory of Transfinite Cardinals developed by Cantor in 1874.

With finite sets there are two natural ways to check if they have the same number of members:

- count the members, call this the cardinality, and see if the two sets have the same cardinality; or
- pair off the members from each set without leftovers (requiring no need to compute the total number of elements in each set).

It is the latter technique that can be used on infinite sets to determine if one set is larger or equal to another in cardinality. The former way, the one we tend to prefer to use, turns out to be only sensible for finite sets.

The basic tools of working with transfinite sets is to either find a clever mapping taking one set to the other (and hence assert that one is "equal" to the other) or establish a contradiction that one set cannot be put in one-to-one correspondence with another (and hence assert that one is "bigger" than the other).

Natural numbers and integers

Consider the three infinite sets

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \tag{1.4a}$$

$$\mathbb{Z}^{\star} = \{0, 1, 2, 3, \ldots\}, \tag{1.4b}$$

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \tag{1.4c}$$

corresponding to the set of all natural numbers, non-negative integers and integers, respectively. Cantor calls the cardinality of the set of all natural numbers $\aleph_0$ (aleph null); see Figure 1.1.[9] It is also called the countably infinite, or denumerable, or denumerably infinite, or equipollent to the ordinal numbers —

[8] This is essential for transfinite taunting.

[9] That is, we could imagine replacing the symbol ∞ with $\aleph_0$ although, fortunately no-one follows this practice.