Cambridge University Press
978-1-107-00854-0 - Linear Algebraic Groups and Finite Groups of Lie Type
Gunter Malle and Donna Testerman
Excerpt
More information

PART I

LINEAR ALGEBRAIC GROUPS

In this part we introduce the main objects of study, linear algebraic groups over algebraically closed fields.

We assume that the reader is familiar with basic concepts and results from commutative algebra and algebraic geometry. More specifically, the reader should know about affine and projective varieties, their associated coordinate ring, their dimension, the Zariski topology, and basic properties thereof.

In Chapter 1 we define our main objects of study. The examples which will guide us throughout the text are certain subgroups or quotient groups of the isometry group of a finite-dimensional vector space equipped with a bilinear or quadratic form. We state the important result which says that any linear algebraic group is a closed subgroup of some group of invertible matrices over our fixed field, which is nearly obvious for all of our examples (the proof will be given in Chapter 5). In Chapter 2, we show that the Jordan decomposition of a matrix results in a uniquely determined Jordan decomposition of elements in a linear algebraic group. This in turn gives us the notion of semisimple and unipotent elements in these groups. We establish the important result that any group consisting entirely of unipotent elements is conjugate to a subgroup of the upper unitriangular matrices.

Chapter 3 is devoted to the structure theory of commutative linear algebraic groups. In particular, Theorem 3.1 focuses attention on groups consisting entirely of unipotent elements or of semisimple elements. Theorem 3.2 classifies the connected one-dimensional linear algebraic groups. While one can say something about the structure of connected commutative groups consisting entirely of unipotent elements, these will not play a role in this text. Hence we turn at this point to commutative groups consisting entirely of semisimple elements and introduce the notion of a torus, its character group and its cocharacter group. These will play a crucial role in the classification of semisimple groups. We turn in Chapter 4 to the structure theory of connected solvable groups, for which the prototype is the group of upper triangular invertible matrices. Indeed, the Lie–Kolchin theorem (Theorem 4.1 and Corollary 4.2) shows that any such group is isomorphic to a closed subgroup of the group of upper triangular matrices. The importance of closed connected solvable subgroups will become apparent in Chapter 6.

But before defining these so-called Borel subgroups, we must extend our theory to cover group actions and in particular quotient groups; this is the content of Chapter 5. The results on homogeneous spaces prepare the terrain for establishing the main result of Chapter 6, the Borel fixed point theorem, Theorem 6.1, some of whose many applications we discuss. We can also finally define the radical of a linear algebraic group and establish its connection with Borel subgroups. In these two chapters, we omit some essential geometric arguments and notions. In particular, we do not prove results on complete varieties but restrict ourselves only to projective varieties.

The last three chapters of this part are devoted to introducing the combinatorial data which classifies semisimple algebraic groups and to establishing structural results and the classification theorem. The most important ingredient of the data is a root system, which is obtained via the adjoint representation of the group, acting on its tangent space; this theory is described in Chapter 7. Theorem 8.17 is the main structural result on reductive groups and we study in detail the case of the group SL_2 in order to sketch a proof of this result. The final chapter describes the classification of semisimple algebraic groups in terms of the data mentioned above. We conclude by explaining where our standard examples appear in this classification.

1

Basic concepts

Throughout, k denotes an algebraically closed field of arbitrary characteristic.

1.1 Linear algebraic groups and morphisms

Recall that a subset X of k^n of the form

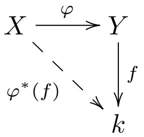
$$X = X(I) = \{(x_1, \ldots, x_n) \in k^n \mid f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}$$

for some ideal $I \lhd k[T_1, \ldots, T_n]$ is called an *algebraic set*. Taking complements of algebraic sets as open sets defines a topology on k^n, the *Zariski topology*.

An *affine algebraic variety* is an algebraic set together with the induced Zariski topology. (We will often omit the word "algebraic".) For $X \subseteq k^n$ an affine algebraic variety, let $I \lhd k[T_1, \ldots, T_n]$ denote the (radical) ideal of polynomials vanishing identically on X. The quotient ring $k[X] = k[T_1, \ldots, T_n]/I$ is called the *coordinate algebra* or *algebra of regular functions on* X since it can be naturally identified with the algebra of all polynomial functions on X with values in k.

If $X \subseteq k^n$, $Y \subseteq k^m$ are affine varieties, their cartesian product $X \times Y$ is naturally an algebraic set in k^{n+m}, hence possesses the structure of an affine variety. We will always consider the product $X \times Y$ equipped with the Zariski topology, not with the product topology, which in general is different. Note that $k[X \times Y] \cong k[X] \otimes_k k[Y]$.

A map $\varphi : X \to Y$ between two affine varieties X, Y, which can be defined by polynomial functions in the coordinates, is called a *morphism* of affine varieties. Note that morphisms are continuous in the Zariski topology. A morphism $\varphi : X \to Y$ induces functorially a k-algebra homomorphism $\varphi^* : k[Y] \to k[X]$ via $\varphi^*(f) := f \circ \varphi$ for $f \in k[Y]$.



In fact, the above defines a contravariant equivalence between the category of affine varieties with morphisms of varieties and the category of finitely generated reduced k-algebras with k-algebra homomorphisms, the so-called affine k-algebras, see [32, §1.5].

We can now define our main object of study.

Definition 1.1 A *linear algebraic group* is an affine algebraic variety equipped with a group structure such that the group operations (multiplication and inversion)

$$\mu : G \times G \longrightarrow G, \qquad (g,h) \longmapsto gh, \qquad\qquad i : G \longrightarrow G, \qquad g \longmapsto g^{-1},$$

are morphisms of varieties. (Recall our convention on the topology on $G \times G$.)

Example 1.2 The base field k provides two natural examples of algebraic groups:

(1) The additive group $G = (k, +)$ of k is defined by the zero ideal $I = (0)$ in $k[T]$, and addition is given by a polynomial; hence G is an algebraic group, with coordinate ring $k[G] = k[T]$. The group G is called the *additive group*, noted $\mathbf{G}_{\mathrm{a}}$.

(2) The multiplicative group $G = (k^\times, \cdot)$ of k can be identified with the set of pairs $\{(x, y) \in k^2 \mid xy = 1\}$ (where multiplication is componentwise, again given by polynomials), which is the algebraic set defined by the ideal $I = (XY - 1) \lhd k[X, Y]$. So here $k[G] = k[X, Y]/(XY - 1) \cong k[X, X^{-1}]$. The group G is called the *multiplicative group* and noted $\mathbf{G}_{\mathrm{m}}$.

It is not immediately obvious from the above definition that the *general linear group*

$$\mathrm{GL}_n := \{A \in k^{n \times n} \mid \det A \neq 0\}$$

of invertible $n \times n$-matrices over k is an algebraic group, since the determinant condition is not a closed condition. But as for $\mathbf{G}_{\mathrm{m}}$ above, GL_n can be identified with the closed subset

$$\{(A, y) \in k^{n \times n} \times k \mid \det A \cdot y = 1\},$$

with componentwise multiplication, via $A \mapsto (A, \det A^{-1})$. Clearly multiplication is a polynomial map, and by Cramer's rule, the same holds for inversion. Thus GL_n is a further (and very important) example of a linear algebraic group. Its coordinate ring is given by

$$\begin{aligned} k[\mathrm{GL}_n] &= k[T_{ij}, Y \mid 1 \le i,j \le n]/(\det(T_{ij})Y - 1) \\ &\cong k[T_{ij} \mid 1 \le i,j \le n]_{\det(T_{ij})}, \end{aligned}$$

the localization of $k[T_{ij} \mid 1 \le i,j \le n]$ at the multiplicative set generated by $\det(T_{ij})$. Note that $\mathrm{GL}_1 = \mathbf{G}_\mathrm{m}$.

Maps between linear algebraic groups should preserve not only the group structure, but also the structure as an affine variety:

Definition 1.3 A map $\varphi : G_1 \to G_2$ of linear algebraic groups is a *morphism of linear algebraic groups* if it is a group homomorphism and also a morphism of varieties, that is, the induced map $\varphi^* : k[G_2] \to k[G_1]$ is a k-algebra homomorphism.

Example 1.4 (1) If $G \le \mathrm{GL}_n$ is a closed subgroup then the natural embedding $G \hookrightarrow \mathrm{GL}_n$ is a morphism of linear algebraic groups.

(2) The determinant map $\det : \mathrm{GL}_n \to \mathbf{G}_\mathrm{m}$, $A \mapsto \det A$, is a group homomorphism and clearly also a morphism of varieties, so a morphism of algebraic groups.

Proposition 1.5 *Kernels and images of morphisms of algebraic groups are closed.*

For the proof of the above statement, we will make use of the following property of morphisms of varieties, which will also be used in subsequent chapters (see [66, Thm. 1.9.5] or [26, Cor. 2.2.8]):

Proposition 1.6 *Let $\varphi : X \to Y$ be a morphism of varieties. Then $\varphi(X)$ contains a non-empty open subset of $\overline{\varphi(X)}$.*

Proof of Proposition 1.5 Let $\varphi : G \to H$ be a morphism of algebraic groups. Since $\ker(\varphi) = \varphi^{-1}(1)$ and φ is a continuous map, $\ker(\varphi)$ is closed. By Proposition 1.6, $\varphi(G)$ contains a non-empty open subset of $\overline{\varphi(G)}$; then $\varphi(G)$ is closed, by Exercise 10.3(d). □

It is clear that any closed subgroup of GL_n inherits the structure of a linear algebraic group. In fact, the converse is also true:

Theorem 1.7 *Let G be a linear algebraic group. Then G can be embedded as a closed subgroup into GL_n for some n.*

The proof of this crucial characterization will be given as a corollary to Theorem 5.5. For example, the map

$$\mathbf{G}_a \longrightarrow \mathrm{GL}_2, \qquad c \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix},$$

defines an embedding of the additive group $\mathbf{G}_a$ as a closed subgroup in GL_2; note that this map is in fact an isomorphism of algebraic groups onto its image.

1.2 Examples of algebraic groups

We introduce some further important examples of linear algebraic groups which will show up throughout the text. We start with three natural subgroups of GL_n. Clearly the group of invertible upper triangular matrices

$$\mathrm{T}_n := \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \text{ invertible} \right\} = \{(a_{ij}) \in \mathrm{GL}_n \mid a_{ij} = 0 \text{ for } i > j\},$$

its subgroup of upper triangular matrices with 1's on the diagonal

$$\mathrm{U}_n := \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} = \{(a_{ij}) \in \mathrm{T}_n \mid a_{ii} = 1 \text{ for } 1 \le i \le n\},$$

and the group of diagonal invertible matrices

$$\mathrm{D}_n := \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \text{ invertible} \right\} = \{\mathrm{diag}(a_1, \ldots, a_n) \mid a_i \neq 0 \text{ for } 1 \le i \le n\},$$

are closed subgroups of GL_n, hence linear algebraic groups.

Recall that a group is called *nilpotent* if its *descending central series* defined by

$$\mathcal{C}^0 G := G, \qquad \mathcal{C}^i G := [\mathcal{C}^{i-1} G, G] \ \text{ for } i \ge 1,$$

eventually reaches 1. It is not hard to see that U_n is a nilpotent group, with $\mathcal{C}^{n-1}(\mathrm{U}_n) = 1$. (One uses the filtration of U_n by normal subgroups $V_m = \{(a_{ij}) \in \mathrm{U}_n \mid a_{ij} = 0 \text{ for } 1 \le j - i \le m\}$, for $1 \le m \le n-1$.)

Furthermore, the *derived series* of a group G is defined by

$$G^{(0)} := G, \qquad G^{(i)} := [G^{(i-1)}, G^{(i-1)}] \ \text{ for } i \ge 1.$$

If there exists some d with $G^{(d)} = 1$, then G is called *solvable*, and the minimal such d is the *derived length* of G. Clearly $G^{(i)} \le \mathcal{C}^i G$, so any nilpotent group is solvable.

In our example, T_n is solvable, with $\mathrm{T}_n^{(1)} = \mathrm{U}_n$. ($\mathrm{U}_n$ is generated by elementary matrices, all of which can be written as commutators.) We will see later on (Corollary 4.2) that T_n is in some sense the prototype of a connected solvable linear algebraic group.

We now define the various families of *classical groups* as groups of isometries of non-degenerate bilinear or quadratic forms on finite-dimensional vector spaces. Recall that k is assumed to be algebraically closed.

The special linear groups

The *special linear group*

$$\mathrm{SL}_n := \left\{(a_{ij}) \in k^{n\times n} \mid \det(a_{ij}) = 1\right\}$$

of $n \times n$-matrices of determinant 1 is a closed subgroup of GL_n, with coordinate ring

$$k[\mathrm{SL}_n] = k[T_{ij} \mid 1 \le i, j \le n]/(\det(T_{ij}) - 1).$$

As k is algebraically closed, we clearly have $\mathrm{GL}_n = Z(\mathrm{GL}_n) \cdot \mathrm{SL}_n$.

The symplectic groups

For $n \ge 1$ let $J_{2n} := \begin{pmatrix} 0 & K_n \\ -K_n & 0 \end{pmatrix}$ where $K_n := \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$. The *symplectic group* in dimension $2n$ is the closed subgroup

$$\mathrm{Sp}_{2n} = \left\{A \in \mathrm{GL}_{2n} \mid A^{\mathrm{tr}} J_{2n} A = J_{2n}\right\}$$

of GL_{2n}; so it is the group of invertible linear transformations of the even-dimensional vector space k^{2n} leaving invariant the non-degenerate skew-symmetric bilinear form with Gram matrix J_{2n} (a so-called *symplectic form*). Here, it is no longer so easy to explicitly write down the coordinate ring.

One can show that Sp_{2n} is generated by transvections (see [79, 8.5]), and hence $\mathrm{Sp}_{2n} \le \mathrm{SL}_{2n}$, and that for $n = 1$, any matrix of determinant 1 is symplectic. So $\mathrm{Sp}_2 = \mathrm{SL}_2$, while for all $n \ge 2$, Sp_{2n} is a proper subgroup of SL_{2n}.

The *conformal symplectic group* is the closed subgroup of GL_{2n} defined as

$$\mathrm{CSp}_{2n} := \left\{A \in \mathrm{GL}_{2n} \mid A^{\mathrm{tr}} J_{2n} A = cJ_{2n} \text{ for some } c \in k^\times\right\},$$

the group of transformations leaving J_{2n} invariant up to a non-zero scalar. It contains Sp_{2n} as a closed normal subgroup.

The odd-dimensional orthogonal groups

First assume that $\mathrm{char}(k) \neq 2$. For $n \geq 1$ the *orthogonal group* in (odd) dimension $2n+1$ is defined by

$$\mathrm{GO}_{2n+1} = \left\{A \in \mathrm{GL}_{2n+1} \mid A^{\mathrm{tr}} K_{2n+1} A = K_{2n+1}\right\}$$

with K_{2n+1} as above. Thus, this is the group of invertible linear transformations leaving invariant the non-degenerate symmetric bilinear form with Gram matrix K_{2n+1}.

If $\mathrm{char}(k) = 2$, skew-symmetric and symmetric bilinear forms coincide, and the previous construction just yields the symplectic group in dimension $2n$. For arbitrary k the orthogonal groups have to be defined using the quadratic form

$$f : k^{2n+1} \longrightarrow k, \quad f(x_1, \ldots, x_{2n+1}) := x_1 x_{2n+1} + x_2 x_{2n} + \cdots + x_n x_{n+2} + x_{n+1}^2,$$

on k^{2n+1} associated to K_{2n+1}. The group of isometries

$$\mathrm{GO}_{2n+1} = \{A \in \mathrm{GL}_{2n+1} \mid f(Ax) = f(x) \text{ for all } x \in k^{2n+1}\}$$

of f is the *odd-dimensional orthogonal group* over k. (For $\mathrm{char}(k) \neq 2$ this defines the same group as before.)

Again there is a conformal version

$$\mathrm{CO}_{2n+1} := \left\{A \in \mathrm{GL}_{2n+1} \mid \exists c \in k^{\times} : f(Ax) = cf(x) \text{ for all } x \in k^{2n+1}\right\},$$

the odd-dimensional *conformal orthogonal group*, containing GO_{2n+1} as a closed normal subgroup.

The even-dimensional orthogonal groups

For even dimension $2n \geq 2$ the orthogonal group is defined using the quadratic form

$$f : k^{2n} \longrightarrow k, \qquad f(x_1, \ldots, x_{2n}) := x_1 x_{2n} + x_2 x_{2n-1} + \cdots + x_n x_{n+1},$$

on k^{2n} associated to K_{2n}. The group of isometries

$$\mathrm{GO}_{2n} = \{A \in \mathrm{GL}_{2n} \mid f(Ax) = f(x) \text{ for all } x \in k^{2n}\}$$

of f is the *even-dimensional orthogonal group* over k. For $\mathrm{char}(k) \neq 2$ we can also obtain this as the group of invertible linear transformations leaving invariant the non-degenerate symmetric bilinear form with Gram matrix K_{2n}:

$$\mathrm{GO}_{2n} = \left\{A \in \mathrm{GL}_{2n} \mid A^{\mathrm{tr}} K_{2n} A = K_{2n}\right\}.$$

The even-dimensional *conformal orthogonal group* is defined as before as

$$\mathrm{CO}_{2n} := \left\{A \in \mathrm{GL}_{2n} \mid \exists c \in k^{\times} : f(Ax) = cf(x) \text{ for all } x \in k^{2n}\right\}.$$

Our choice of symmetric, skew-symmetric and quadratic forms above may seem a bit arbitrary. In fact, any non-degenerate symmetric bilinear form leads to the same group up to conjugacy, and similarly for non-degenerate skew-symmetric bilinear forms, respectively quadratic forms (see for example [2, §7]), but for the choices made above, certain natural subgroups have a particularly nice shape, as will become apparent later.

As a final example, let G be a finite group. Then G has a faithful permutation representation, that is, there is an embedding $G \hookrightarrow \mathfrak{S}_n$ into a *symmetric group* $\mathfrak{S}_n$ for some n. Moreover, $\mathfrak{S}_n \hookrightarrow \mathrm{GL}_n$ via the natural permutation representation. Combining these two homomorphisms we get an embedding $G \hookrightarrow \mathrm{GL}_n$ whose image is a closed subgroup (i.e., the set of zeros of a finite set of polynomial functions). Therefore, any finite group can be considered as a linear algebraic group, with the discrete topology.

1.3 Connectedness

We now recall a topological notion which will play a crucial role in the study of linear algebraic groups.

Definition 1.8 A topological space X is called *irreducible* if it cannot be decomposed as $X = X_1 \cup X_2$ where X_i is a non-empty proper closed subset for $i = 1, 2$.

In view of the importance of this concept, we present some further elementary characterizations of irreducibility:

Proposition 1.9 *The following are equivalent for an affine algebraic variety X:*

(i) *X is irreducible.*
(ii) *Every non-empty open subset of X is dense.*
(iii) *Any two non-empty open subsets of X intersect non-trivially.*
(iv) *The vanishing ideal I of X is a prime ideal.*
(v) *$k[X]$ is an integral domain.*

Proof (i)⇔(ii): Indeed, if $X_1 \subseteq X$ is open then $X = \overline{X}_1 \cup (X \setminus X_1)$. Next, (ii)⇔(iii) is obvious. The equivalence (i)⇔(iv) is shown in [32, Prop. 1.3C]. Finally, the equivalence of (iv) and (v) is well known. □

Furthermore, we need the following basic properties (see [32, Prop. 1.3A, 1.3B and 1.4] and also Exercise 10.1):

Cambridge University Press
978-1-107-00854-0 - Linear Algebraic Groups and Finite Groups of Lie Type
Gunter Malle and Donna Testerman
Excerpt
More information

Proposition 1.10 *Let X, Y be affine varieties. Then we have:*

(a) *A subset Z of X is irreducible if and only if its closure $\overline{Z}$ is irreducible.*
(b) *If X is irreducible and $\varphi : X \to Y$ is a morphism then $\varphi(X)$ is irreducible.*
(c) *If X, Y are irreducible then $X \times Y$ is irreducible.*
(d) *X has only finitely many maximal irreducible subsets X_i, and $X = \bigcup X_i$. In other words, every variety is a finite union of its maximal irreducible subsets.*

The maximal irreducible subsets in the preceding statement are called the *irreducible components* of X. Note that by (a) irreducible components are necessarily closed.

Definition 1.11 A topological space X is said to be *connected* if it cannot be decomposed as a disjoint union $X = X_1 \sqcup X_2$, where the X_i's are non-empty closed subsets.

Note that any irreducible set is connected; the converse is not true in general. See Exercise 10.2.

Example 1.12 Let's next look at some linear algebraic groups.

(1) $\mathbf{G}_\mathrm{a}$ and $\mathbf{G}_\mathrm{m}$ are connected by Proposition 1.9(v) since $k[\mathbf{G}_\mathrm{a}] = k[T]$ and $k[\mathbf{G}_\mathrm{m}] = k[T, T^{-1}]$ are integral domains.
(2) GL_n is connected since $k[\mathrm{GL}_n] = k[T_{ij}]_{\det(T_{ij})}$ is an integral domain, being a localization of the polynomial ring $k[T_{ij}]$.

The next result gives a first example of how the Zariski topology on a linear algebraic group allows one to deduce group theoretic structural results.

Proposition 1.13 *Let G be a linear algebraic group.*

(a) *The irreducible components of G are pairwise disjoint, so they are the connected components of G.*
(b) *The irreducible component G° containing $1 \in G$ is a closed normal subgroup of finite index in G.*
(c) *Any closed subgroup of G of finite index contains G°.*

Proof (a) Let X, Y be two irreducible components of G. Assume that $g \in X \cap Y$. Since multiplication by g^{-1} is a morphism of G onto itself, $g^{-1}X, g^{-1}Y$ are irreducible by Proposition 1.10(b) and $1 \in g^{-1}X \cap g^{-1}Y$. Therefore, without loss of generality we may assume that $1 \in X \cap Y$. Now, $\mu(X \times Y) = XY$ is irreducible by Proposition 1.10(b), (c). As $X = X \cdot 1 \subseteq XY$ and