A Course on Set Theory

Set theory is the mathematics of infinity and part of the core curriculum for mathematics majors. This book blends theory and connections with other parts of mathematics so that readers can understand the place of set theory within the wider context. Beginning with the theoretical fundamentals, the author proceeds to illustrate applications to topology, analysis and combinatorics, as well as to pure set theory. Concepts such as Boolean algebras, trees, games, dense linear orderings, ideals, filters and club and stationary sets are also developed.

Pitched specifically at undergraduate students, the approach is neither esoteric nor encyclopedic. The author, an experienced instructor, includes motivating examples and over 100 exercises designed for homework assignments, reviews and exams. It is appropriate for undergraduates as a course textbook or for self-study. Graduate students and researchers will also find it useful as a refresher or to solidify their understanding of basic set theory.

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Note to the instructor

This book was written for an undergraduate set theory course, which is taught at Carnegie Mellon University every spring. It is aimed at serious students who have taken at least one proof-based mathematics course in any area. Most are mathematics or computer science majors, or both, but life and physical science, engineering, economics and philosophy students have also done well in the course. Other students have used this book to learn the material on their own or as a refresher. Mastering this book and learning a bit of mathematical logic, which is not included, would prepare the student for a first-year graduate level set theory course in the future. The book also contains the minimum amount of set theory that everyone planning to go on in math should know.

I have included slightly more than the maximum amount of material that I have covered in a fifteen-week semester. But I do not reach the maximum every time; in fact, only once. For a slower pace or shorter academic term, one of several options would be to skip Sections 5.6 and 7.2, which are more advanced.

There are over one hundred exercises, more than enough for eight homework assignments, two midterm exams, a final exam and review problems before each exam. Exercises are located at the ends of Chapters 1, 2, 3, 4 and 6. They are also dispersed throughout Chapters 5 and 7. This slight lack of uniformity is tied to the presentation and ultimately makes sense.

In roughly the first half of the book, through Chapter 4, I develop ordinal and cardinal arithmetic starting from the axioms of Zermelo–Fraenkel Set Theory with the Axiom of Choice (ZFC). In other words, this is not a book on what some call *naive set theory*. There is one minor way in which the presentation is not entirely rigorous. Namely, in listing the axioms of ZFC, I use the imprecise word *property* instead of the formal expression *first-order formula* because mathematical logic is not a prerequisite for the course.

Some other textbooks develop the theory of cardinality for as long as possible without using the Axiom of Choice (AC). I do not take this approach because it adds technicalities, which are not used later in the course, and gives students the misleading impression that AC is controversial. By assuming AC from the start, I am able to streamline the theory of cardinality. I may note how AC has been used in a proof but I do not belabor the point. Once, when an alternate proof without AC exists, it is outlined in an exercise.

The second half of the book is designed to give students a sense of the place of set theory within mathematics. Where I draw connections to other fields, I include all the necessary background material. Some of the other areas that come up in Chapter 5 are topology, metric spaces, trees, games and Ramsey theory. The real numbers are constructed using Dedekind cuts in Chapter 6. Chapter 7 introduces the student to filters and ideals, and takes up the combinatorics of uncountable sets. There is no section specifically on Boolean algebra but it is one of the recurring themes in the exercises throughout the book. For the reader's convenience, I have briefly summarized the results on Boolean algebra in the Appendix. All of this material is self-contained.

As I mentioned, before starting this book, students should have at least one semester's worth of experience reading and writing proofs in any area of mathematics; it does not matter which area. They should be comfortable with sets, relations and functions, having seen and used them at a basic level earlier. They should know the difference between integers, rational numbers and real numbers, even if they have not seen them explicitly constructed. And they should have experience with recursive definitions along the integers and proofs by induction on the integers. These notions come up again here but in more sophisticated ways than in a first theoretical mathematics course. There are no other prerequisites. However, because of the emphasis on connections to other fields, students who have taken courses on logic, analysis, algebra, or discrete mathematics will enjoy seeing how set theory and these other subjects fit together. The unifying perspective of set theory will give students significant advantages in their future mathematics courses.

Acknowledgements

As an undergraduate, I studied from *Elements of set theory* by Herbert Enderton and *Set theory: an introduction to independence proofs* by Kenneth Kunen. When I started teaching undergraduate set theory, I recommended *Introduction to set theory* by Karel Hrbacek and Thomas Jech to my students. The reader who knows these other textbooks will be aware of their positive influence.

This book began as a series of handouts for undergraduate students at Carnegie Mellon University. Over the years, they found typographical errors and indicated what needed more explanation, for which I am grateful. I also thank Michael Klipper for proofreading a draft of the book in Spring 2008, when he was a graduate student in the CMU Doctor of Philosophy program.

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