Stochastic Processes

This comprehensive guide to stochastic processes gives a complete overview of the theory and addresses the most important applications. Pitched at a level accessible to beginning graduate students and researchers from applied disciplines, it is both a course book and a rich resource for individual readers. Subjects covered include Brownian motion, stochastic calculus, stochastic differential equations, Markov processes, weak convergence of processes, and semigroup theory. Applications include the Black–Scholes formula for the pricing of derivatives in financial mathematics, the Kalman–Bucy filter used in the US space program, and also theoretical applications to partial differential equations and analysis. Short, readable chapters aim for clarity rather than for full generality. More than 350 exercises are included to help readers put their new-found knowledge to the test and to prepare them for tackling the research literature.

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To Meredith, as always

Contents

Preface

Why study stochastic processes? This branch of probability theory offers sophisticated theorems and proofs, such as the existence of Brownian motion, the Doob–Meyer decomposition, and the Kolmogorov continuity criterion. At the same time stochastic processes also have far-reaching applications: the explosive growth in options and derivatives in financial markets throughout the world derives from the Black–Scholes formula, while NASA relies on the Kalman–Bucy method to filter signals from satellites and probes sent into outer space.

A graduate student taking a year-long course in probability theory first learns about sequences of random variables and topics such as laws of large numbers, central limit theorems, and discrete time martingales. In the second half of the course, the student will then turn to stochastic processes, which is the subject of this text. Topics covered here are Brownian motion, stochastic integrals, stochastic differential equations, Markov processes, the Black–Scholes formula of financial mathematics, the Kalman–Bucy filter, as well as many more.

The 42 chapters of this book can be grouped into seven parts. The first part consists of Chapters 1–8, where some of the basic processes and ideas are introduced, including Brownian motion. The next group of chapters, Chapters 9–15, introduce the theory of stochastic calculus, including stochastic integrals and Itô's formula. Chapters 16–18 explore jump processes. This requires a study of the foundations of stochastic processes, which is also known as the general theory of processes. Next we take up Markov processes in Chapters 19–23. A formidable obstacle to the study of Markov processes is the notation, and I have attempted to make this as accessible as possible. Chapters 24–29 involve stochastic differential equations. Two very important applications, to financial mathematics and to filtering, appear in Chapters 28 and 29, respectively. Probability measures on metric spaces and the weak convergence of random variables taking values in a metric space prove to be relevant to the study of stochastic processes. These and related topics are treated in Chapters 30–35. We then return to Markov processes, namely, their construction and some important examples, in Chapters 36–42. Tools used in the construction include infinitesimal generators, Dirichlet forms, and solutions to stochastic differential equations, while two important examples that we consider are diffusions on the real line and Lévy processes.

The prerequisites to this book are a sound knowledge of basic measure theory and a course in the classical aspects of probability. The probability topics needed are provided (with proofs) in an appendix.

There is far too much material in this book to cover in a single semester, and even too much for a full year. I recommend that as a minimum the following chapters be studied: Chapters 1–5, Chapters 9–13, Chapters 19–21, and Chapter 24. If possible, include either

Chapter 28 or Chapter 29. In Chapter 11, the statement and corollaries of Itô's formula are very important, but the proof of Itô's formula may be omitted.

I would like to thank the many students who patiently sat through my lectures, pointed out errors, and made suggestions. I especially would like to thank my colleague Sasha Teplyaev who taught a course from a preliminary version of this book and made a great number of useful suggestions.

Frequently used notation

Here are some notational conventions we will use. We use the letter c, either with or without subscripts, to denote a finite positive constant whose exact value is unimportant and which may change from line to line. We use $B(x, r)$ to denote the open Euclidean ball centered at x with radius r. $a \wedge b$ is the minimum of a and b, while $a \vee b$ is the maximum of a and b. $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. The symbol $\exists$ is used in a few formulas and means "there exists." $\mathbb{Q}$, $\mathbb{Q}_+$, $\mathbb{N}$, and $\mathbb{Z}$ denote the rationals, the positive rationals, the natural numbers, and the integers, respectively. If C is a matrix, C^{T} is the transpose of C.

For a set A, we use A^c for the complement of A. If A is a subset of a topological space, $\overline{A}$, A^0, and ∂A denote the closure, interior, and boundary of A, respectively.

Given a topological space $\mathcal{S}$, we use $C(\mathcal{S})$ for the space of continuous functions on $\mathcal{S}$, where we use the supremum norm. If $\mathcal{S}$ is a domain in $\mathbb{R}^d$, $C^k(\mathcal{S})$ refers to the set of continuous functions with domain $\mathcal{S}$ whose partial derivatives up to order k are continuous. C^∞ functions are those that are infinitely differentiable.

We will on a few occasions use the Fourier transform, which we define by

$$\widehat{f}(u) = \int e^{iu \cdot x} f(x)\, dx$$

for f integrable. This agrees with the convention in Rudin (1987).

If X is a stochastic process whose paths are right continuous with left limits, then $X_{t-} = \lim_{s<t, s\to t} X_s$ and $\Delta X_t = X_t - X_{t-}$.