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978-1-107-00799-4 - Automorphic Representations and L-Functions for the General Linear Group, Volume II

Dorian Goldfeld and Joseph Hundley

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