12

The classical theory of automorphic forms for $GL(n, \mathbb{R})$

In this chapter we present a classical description of the theory of automorphic forms of arbitrary weight and level for the group $GL(n, \mathbb{R})$. This generalizes the theory of K_∞-fixed automorphic forms presented in [Goldfeld, 2006]. For the reader who is mainly interested in the theory of automorphic representations, this chapter may be skipped on a first reading.

12.1 Iwasawa decomposition for $GL(n, \mathbb{R})$

Definition 12.1.1 (Generalized upper half plane) Let $n \geq 2$. The generalized upper half plane $\mathfrak{h}^n$ associated to $GL(n, \mathbb{R})$ is defined to be the set of all $n \times n$ matrices of the form $z = x \cdot y$ where

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y'_{n-1} & & & & \\ & y'_{n-2} & & & \\ & & \ddots & & \\ & & & y'_1 & \\ & & & & 1 \end{pmatrix},$$

with $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y'_i > 0$ for $1 \leq i \leq n-1$.

To simplify later formulae and notation in this book, we will always express y in the form:

$$y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

with $y_i > 0$ for $1 \leq i \leq n-1$. Note that this can always be done since $y'_i \neq 0$ for $1 \leq i \leq n-1$.

Cambridge University Press
978-1-107-00799-4 - Automorphic Representations and L-Functions for the General Linear Group, Volume II
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Proposition 12.1.2 (The Iwasawa decomposition for $GL(n,\mathbb{R})$) *Let $n \geq 2$. Every matrix $g \in GL(n,\mathbb{R})$ has a factorization of the form*

$$g = \widetilde{g} \cdot d \cdot k,$$

with $\widetilde{g} \in \mathfrak{h}^n$, uniquely determined, $k \in O(n,\mathbb{R})$, and d a non-zero diagonal matrix in the center of $GL(n,\mathbb{R})$. Furthermore, k and d are also uniquely determined up to multiplication by $\pm I_n$ where I_n is the $n \times n$ identity matrix.

Proof This is Proposition 1.2.6 of [Goldfeld, 2006]. □

Definition 12.1.3 (Action of $GL(n,\mathbb{R})$ on $\mathfrak{h}^n$) For $g \in GL(n,\mathbb{R})$ and $z \in \mathfrak{h}^n$ define

$$g\,.\,z := \widetilde{g \cdot z} \qquad (\forall g \in GL(n,\mathbb{R}),\ z \in \mathfrak{h}^n).$$

Here, the product on the right hand side is matrix multiplication, and $\widetilde{g \cdot z} \in \mathfrak{h}^n$ is the $\mathfrak{h}^n$ component of the Iwasawa decomposition of $g \cdot z$, as in Proposition 12.1.2.

12.2 Congruence subgroups of $SL(n,\mathbb{Z})$

Definition 12.2.1 (Principal congruence subgroup of $SL(n,\mathbb{Z})$) Let $n \geq 2$. Fix an integer $N \geq 1$. A principal congruence subgroup of $SL(n,\mathbb{Z})$ of level N is the kernel of the map

$$SL(n,\mathbb{Z}) \to SL(n,\mathbb{Z}/N\mathbb{Z}).$$

We denote the principal congruence subgroup of $SL(n,\mathbb{Z})$ of level N by $\Gamma(N)$.

Definition 12.2.2 (Congruence subgroup of $SL(n,\mathbb{Z})$) Let $n \geq 2$. A subgroup $\Gamma \subset SL(n,\mathbb{Z})$ is called a congruence subgroup if Γ contains $\Gamma(N)$ for some $N \geq 1$. The least such N is called the level of Γ.

When $n = 2$, it had been known for a long time that there exist infinitely many subgroups of finite index in $SL(2,\mathbb{Z})$ which are not congruence subgroups [Magnus, 1974]. It was proved independently in [Bass-Lazard-Serre, 1964], [Mennicke, 1965] that every subgroup of finite index in $SL(n,\mathbb{Z})$ (with $n \geq 3$) must be a congruence subgroup. This was further generalized in [Bass-Milnor-Serre, 1967]. The congruence subgroup problem for other semisimple algebraic groups is an area of active research. See [Ragunathan, 2004] for a survey.

Definition 12.2.3 (The congruence subgroup $\Gamma_0(N)$) Let $n \geq 2$. For an integer $N \geq 2$, the congruence subgroup $\Gamma_0(N)$ is defined to be the multiplicative group of all matrices of determinant 1 which are of the form

$$\left\{ \begin{pmatrix} A & B \\ C & d \end{pmatrix} \in SL(n,\mathbb{Z}) \;\middle|\; \begin{matrix} A \in \mathrm{Mat}(n-1,\ \mathbb{Z}), & B \in \mathrm{Mat}((n-1)\times 1,\ \mathbb{Z}), \\ C \in \mathrm{Mat}(1\times(n-1),\ N\cdot\mathbb{Z}), & d \in \mathbb{Z} \end{matrix} \right\}.$$

Here, for an arbitrary ring R, we define $\mathrm{Mat}(i, R)$, $\mathrm{Mat}(i \times j, R)$ (respectively) to denote the set of all $i \times i$, $i \times j$ (respectively) matrices with coefficients in R. In addition, we define $\Gamma_0(1) := SL(n, \mathbb{Z})$.

Definition 12.2.4 (The character $\widetilde{\chi}$ of $\Gamma_0(N)$) Let $n \geq 2$. Fix an integer $N \geq 1$, and let $\Gamma_0(N)$ be as in Definition 12.2.3. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character $\pmod N$. Then for $\gamma = \left(\begin{smallmatrix} A & B \\ C & d \end{smallmatrix}\right) \in \Gamma_0(N)$ as in Definition 12.2.3, we define $\widetilde{\chi}(\gamma) := \chi(d)$.

12.3 Automorphic functions of arbitrary weight, level, and character

Let $GL(n, \mathbb{R})^+$ denote the subgroup of $GL(n, \mathbb{R})$ consisting of those elements of $GL(n, \mathbb{R})$ with positive determinant. The group $GL(n, \mathbb{R})^+$ acts on the generalized upper half-plane $\mathfrak{h}^n$ as in Definition 12.1.3. This action determines a function

$$\kappa : GL(n, \mathbb{R})^+ \times \mathfrak{h}^n \to SO(n, \mathbb{R})$$

as follows.

Definition 12.3.1 (The function κ) Let $n \geq 2$. For any $\gamma \in GL(n, \mathbb{R})^+$ and any $z = xy \in \mathfrak{h}^n$, as in Definition 12.1.1, there exists a unique $\kappa(\gamma, z) \in SO(n, \mathbb{R})$ and a unique $\widetilde{\gamma z} \in \mathfrak{h}^n$ such that

$$\boxed{\gamma z \;=\; \widetilde{\gamma z} \cdot \kappa(\gamma, z) \cdot d}$$

where $d = \delta I_n$ with $\delta > 0$. Here I_n is the $n \times n$ identity matrix.

Remark The uniqueness of $\widetilde{\gamma z}$ and $\kappa(\gamma, z)$ follows immediately from the Iwasawa decomposition as given in Proposition 12.1.2 .

Example 12.3.2 (The function κ for $n = 2$) Fix $n = 2$. Let $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{h}^2$ and let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$. Then we may explicitly write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{cx+d}{|cz+d|} & \frac{-cy}{|cz+d|} \\ \frac{cy}{|cz+d|} & \frac{cx+d}{|cz+d|} \end{pmatrix}\begin{pmatrix} |cz+d| & 0 \\ 0 & |cz+d| \end{pmatrix},$$

with $z = x + iy$ and $\left(\begin{smallmatrix} y' & x' \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{h}^2$. It follows that

$$\kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} \frac{cx+d}{|cz+d|} & \frac{-cy}{|cz+d|} \\ \frac{cy}{|cz+d|} & \frac{cx+d}{|cz+d|} \end{pmatrix}.$$

Proposition 12.3.3 (The function κ is a one-cocycle) *Let $n \geq 2$. Recall that if $g \in GL(n, \mathbb{R})$, then $\widetilde{g} \in \mathfrak{h}^n$ is uniquely determined in Definition 12.1.2.*

Consider $\kappa : GL(n,\mathbb{R})^+ \times \mathfrak{h}^n \to SO(n,\mathbb{R})$ *as defined in Definition 12.3.1. Then* κ *satisfies the following one-cocycle relation:*

$$\kappa(\gamma\gamma', z) = \kappa\left(\gamma, \widetilde{\gamma' z}\right)\cdot\kappa(\gamma', z)$$

for all $\gamma, \gamma' \in GL(n,\mathbb{R})^+$ *and all* $z \in \mathfrak{h}^n$.

Proof It follows from Definition 12.3.1 that

$$(\gamma\gamma')z = \widetilde{\gamma\gamma' z}\cdot\kappa(\gamma\gamma', z)\cdot d,$$

with unique matrices $\widetilde{\gamma\gamma' z} \in \mathfrak{h}^n$ and $d = \delta I_n$ with $\delta > 0$.

On the other hand, we again have from Definition 12.3.1 that

$$\begin{aligned}\gamma(\gamma' z) &= \gamma\cdot\left(\widetilde{\gamma' z}\cdot\kappa(\gamma', z)\cdot d'\right), \qquad (d' = \delta' I_n \text{ with } \delta' > 0),\\ &= \widetilde{\gamma\cdot\widetilde{\gamma' z}}\cdot\kappa\left(\gamma, \widetilde{\gamma' z}\right)\cdot\kappa(\gamma', z)\cdot d'', \qquad (d'' = \delta'' I_n \text{ with } \delta'' > 0).\end{aligned}$$

It immediately follows from the uniqueness of the Iwasawa decomposition that

$$\widetilde{\gamma\cdot\widetilde{\gamma' z}} = \widetilde{\gamma\gamma' z}$$

and

$$\kappa(\gamma\gamma', z) = \kappa\left(\gamma, \widetilde{\gamma' z}\right)\cdot\kappa(\gamma', z). \quad \square$$

Following Section 3.3 and (3.5.1), one may define automorphic functions of weight $k \in \mathbb{Z}$ for $SL(2,\mathbb{Z})$ as functions $f : \mathfrak{h}^2 \to \mathbb{C}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{cz+d}{|cz+d|}\right)^k f(z)$$

for all $z = \left(\begin{smallmatrix} y & x\\ 0 & 1\end{smallmatrix}\right) i = x + iy$ with $\left(\begin{smallmatrix} y & x\\ 0 & 1\end{smallmatrix}\right) \in \mathfrak{h}^2$. The function

$$J_k(\gamma, z) := \left(\frac{cz+d}{|cz+d|}\right)^k, \qquad \left(\forall\, \gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL(n,\mathbb{R})\right),$$

is a one cocycle as in Example 3.1.4.

One would like to generalize the notion of "weight k" to the higher rank situation of $GL(n,\mathbb{R})$ with $n > 2$. It will turn out that if $n > 2$ then the weight k may be realized as a finite dimensional irreducible representation ρ of $SO(n,\mathbb{R})$. We now proceed to develop these concepts. It will be clear that many of the definitions make sense and many of the results hold without the assumption that ρ is irreducible. However, there is no real loss of generality in assuming that ρ is irreducible, as Exercise 12.11(b) shows, and the requirement of irreducibility maintains the analogy with the classical theory for $GL(2,\mathbb{R})$, where an integral weight k corresponds to an irreducible representation of $SO(2,\mathbb{R})$.

Definition 12.3.4 (The function J_ρ) Let $n \geq 2$ and $r \geq 1$ be integers. Let $\rho : SO(n,\mathbb{R}) \to GL(r,\mathbb{C})$ be an irreducible representation as in Definition 2.5.1. We define a function $J_\rho : GL(n,\mathbb{R})^+ \times \mathfrak{h}^n \to GL(r,\mathbb{C})$ as follows. Let $\gamma \in GL(n,\mathbb{R})^+$ and $z \in \mathfrak{h}^n$. Then we define

$$J_\rho(\gamma, z) \;:=\; \rho\Big(\kappa(\gamma, z)^{-1}\Big),$$

where κ is given by Definition 12.3.1.

Proposition 12.3.5 (The function J_ρ is a one-cocycle) *Let $n \geq 2$. Recall that if $g \in GL(n,\mathbb{R})$ then $\widetilde{g} \in \mathfrak{h}^n$ is uniquely determined in Definition 12.1.2. Consider the function $J_\rho : GL(n,\mathbb{R})^+ \times \mathfrak{h}^n \to GL(r,\mathbb{C})$ defined in Definition 12.3.4. Then J_ρ satisfies the one-cocycle relation*

$$J_\rho(\gamma\gamma', z) \;=\; J_\rho(\gamma', z)\, J_\rho(\gamma, \widetilde{\gamma' z})$$

for all $\gamma, \gamma' \in GL(n,\mathbb{R})^+$ and all $z \in \mathfrak{h}^n$.

Proof It follows immediately from Definition 12.3.4 and Proposition 12.3.3 that

$$\begin{aligned} J_\rho(\gamma\gamma', z) &= \rho\left(\kappa(\gamma\gamma', z)^{-1}\right) \\ &= \rho\left(\kappa(\gamma', z)^{-1} \cdot \kappa\left(\gamma, \widetilde{\gamma' z}\right)^{-1}\right) \\ &= \rho\left(\kappa(\gamma', z)^{-1}\right) \cdot \rho\left(\kappa\left(\gamma, \widetilde{\gamma' z}\right)^{-1}\right) \\ &= J_\rho(\gamma', z)\, J_\rho(\gamma, \widetilde{\gamma' z}). \qquad \square \end{aligned}$$

Remark The one-cocycle J_ρ is the generalization of the classical j-cocycle which appeared in Example 3.1.4.

Example 12.3.6 (The function J_ρ for $n = 2$) For every $k \in \mathbb{Z}$ we may define a one-dimensional representation $\rho_k : SO(2,\mathbb{R}) \to \mathbb{C}^\times$ by letting

$$\rho_k\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) := (\cos\theta + i\sin\theta)^k, \qquad (\theta \in \mathbb{R}).$$

It then follows from Example 12.3.2 that for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ and all $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{h}^2$ that

$$\begin{aligned} \rho_k\left(\kappa\left(\gamma, \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)^{-1}\right) &= \rho_k\left(\begin{pmatrix} \frac{cx+d}{|cz+d|} & \frac{cy}{|cz+d|} \\ \frac{-cy}{|cz+d|} & \frac{cx+d}{|cz+d|} \end{pmatrix}\right) \\ &= \left(\frac{cz+d}{|cz+d|}\right)^k, \qquad (z = x + iy). \end{aligned}$$

With Example 12.3.6 in mind, it should now start to become clear to the reader how to generalize the classical automorphic forms of weight k, studied in Chapter 3, to the higher rank situation of $GL(n,\mathbb{R})$ with $n \geq 3$. The precise definition will be given shortly. It will first be necessary, however, to introduce the analogue of the classical slash operator as given in Definition 3.5.6.

Fix $n \geq 2,\ r \geq 1$, and let $\Phi : \mathfrak{h}^n \to \mathbb{C}^r$ be a smooth vector valued function given by

$$\Phi(z) := \begin{pmatrix} \phi_1(z) \\ \vdots \\ \phi_r(z) \end{pmatrix}, \qquad (z \in \mathfrak{h}^n), \tag{12.3.7}$$

where each $\phi_i : \mathfrak{h}^n \to \mathbb{C},\ (1 \leq i \leq r)$ is smooth.

Definition 12.3.8 (Slash operator) Fix $n \geq 2,\ r \geq 1$, and let $\Phi : \mathfrak{h}^n \to \mathbb{C}^r$ be a smooth vector valued function as in (12.3.7). Let $\rho : SO(n,\mathbb{R}) \to GL(r,\mathbb{C})$ be an irreducible representation. For $z \in \mathfrak{h}^n$ and $\gamma \in GL(n,\mathbb{R})^+$, we define the slash operator $|_\rho$ by

$$\left(\Phi\ \big|_\rho\ \gamma\right)(z) := J_\rho(\gamma, z)^{-1} \cdot \Phi(\gamma\,.\,z),$$

where J_ρ is defined in Definition 12.3.4.

We leave it as an exercise for the reader to show that the slash operator satisfies $\Phi\,\big|_\rho\,\gamma\gamma' = \Phi\,\big|_\rho\,\gamma\,\big|_\rho\,\gamma'$ for all $\gamma, \gamma' \in GL(n,\mathbb{R})^+$.

Definition 12.3.9 (Vector valued automorphic function of weight ρ, level N, character χ) Let $n \geq 2$, $r \geq 1$ be integers. Fix an irreducible representation $\rho : SO(n,\mathbb{R}) \to GL(r,\mathbb{C})$, an integer $N \geq 1$, and a Dirichlet character $\chi \pmod N$. Let $\Gamma_0(N)$ be defined as in Definition 12.2.3 and let $\widetilde{\chi}$ be as in Definition 12.2.4.

A vector valued automorphic function of weight ρ, level N, and character χ is a smooth function $\Phi : \mathfrak{h}^n \to \mathbb{C}^r$, as in (12.3.7), which satisfies the automorphy relation

$$\left(\Phi\,\big|_\rho\,\gamma\right)(z) \ = \ \widetilde{\chi}(\gamma)\,\Phi(z),$$

for all $\gamma \in \Gamma_0(N),\ z \in \mathfrak{h}^n$, and if N is the least integer with this property.

Remark If the representation ρ is not smooth, then the space of vector valued automorphic functions of weight ρ (and any level and character) is trivial. We do not need to exclude non-smooth representations from consideration. However, the reader may always assume that ρ is smooth in what follows. The representations ρ_k considered in Example 12.3.6 are not the only irreducible representations of $SO(2,\mathbb{R})$. They are, however, the only *smooth* irreducible representations of $SO(2,\mathbb{R})$.

In order to define automorphic forms (also called Maass forms) on $\mathfrak{h}^n$ it is necessary to develop the $GL(n)$ versions of the conditions (introduced in Section 3.5) which separate automorphic forms from other automorphic functions. The first step is to generalize Definition 3.3.3 (moderate growth).

Definition 12.3.10 (Moderate growth) A smooth function $\mathfrak{h}^n \to \mathbb{C}$ is said to have moderate growth if, for each fixed $\sigma \in GL(n, \mathbb{Q})$, there exist constants c, C and B such that

$$|f(\sigma \cdot z)|_{\mathbb{C}} \le C \cdot (y_1 \cdot y_2 \cdot \ldots \cdot y_{n-1})^B$$

for all $z = x \cdot y \in \mathfrak{h}^n$, with x, y as in Definition 12.1.1, such that $\min(y_1, \ldots, y_{n-1}) \ge c$.

Here $||_{\mathbb{C}}$ is the usual absolute value on $\mathbb{C}$.

Remarks Suppose $\beta \in GL(n, \mathbb{Q})$ is upper triangular and $z = x \cdot y \in \mathfrak{h}^n$. Write $\beta \cdot z = \widetilde{\beta \cdot z} \cdot \kappa(\beta, z) \cdot d$. Then each diagonal entry of $\widetilde{\beta \cdot z}$ is equal to the corresponding entry of z, times the absolute value of the corresponding entry of β. It follows that in Definition 12.3.10 we may restrict σ to a set of representatives for $GL(n, \mathbb{Q})/B_n(\mathbb{Q})$, where $B_n(\mathbb{Q})$ denotes the group of invertible upper-triangular $n \times n$ matrices with entries in $\mathbb{Q}$. When $n = 2$, this is the set $GL(2, \mathbb{Q})/B_2(\mathbb{Q})$, in natural one-to-one correspondence with the set $\mathbb{Q} \cup \{\infty\}$ of cusps.

The next step is to generalize the action of the weight k Laplacian, defined in Definition 3.5.3. In Chapter 4, it was shown that a classical Maass form f of weight k may be lifted to a function $\widetilde{f} : GL(2, \mathbb{R})^+ \to \mathbb{C}$ satisfying

$$\widetilde{f}\left(g\begin{pmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{pmatrix}\right) = e^{ik\theta'}\widetilde{f}(g), \qquad \left(\forall\, \theta' \in \mathbb{R},\ g \in GL(2, \mathbb{R})^+\right). \tag{12.3.11}$$

In the same chapter, the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$ and its center $Z(U(\mathfrak{g}))$ were introduced. It follows from the proof of (5.5.14) that the weight k Laplacian Δ_k is induced by the action of certain elements of the universal enveloping algebra. (To be precise, $-\frac{1}{2}$ times the Casimir element, or any other element of $Z(U(\mathfrak{g}))$ which is equivalent to this one modulo the ideal generated by the operator D_Z, which kills $\widetilde{f}$ for any f.) This discussion motivates the following definition.

Definition 12.3.12 (Lift of an automorphic function to $GL(n, \mathbb{R})^+$) Let $\Phi : \mathfrak{h}^n \to \mathbb{C}^r$ be a vector valued automorphic function of weight ρ, level N, and character χ as in Definition 12.3.9. Let $GL(n, \mathbb{R})^+$ denote the subgroup of $GL(n, \mathbb{R})$ consisting of all $n \times n$ invertible real matrices with positive determinant.

Define $\widetilde{\Phi} : GL(n,\mathbb{R})^+ \to \mathbb{C}^r$ by

$$\widetilde{\Phi}(g) := \Big(\Phi\mid_\rho g\Big)(I_n), \qquad (g \in GL(n,\mathbb{R})^+),$$

where I_n denotes the $n \times n$ identity matrix, regarded as an element of $\mathfrak{h}^n$.

It follows from the definitions that $\widetilde{\Phi}$ satisfies the following natural generalization of (12.3.11):

$$\widetilde{\Phi}(gk) = \rho(k) \cdot \widetilde{\Phi}(g), \qquad \big(\forall g \in GL(n,\mathbb{R})^+,\ k \in SO(n,\mathbb{R})\big). \qquad (12.3.13)$$

Next, it is necessary to introduce the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} := \mathfrak{gl}(n,\mathbb{C})$, as well as its center $Z(U(\mathfrak{g}))$. We follow Section 2.3 in [Goldfeld, 2006].

Definition 12.3.14 (Universal enveloping algebra and its center) For each $\alpha \in \mathfrak{gl}(n,\mathbb{R})$ we define a differential operator D_α acting on smooth functions $\phi : GL(n,\mathbb{R}) \to \mathbb{C}$ by the formula

$$D_\alpha\phi(g) = \lim_{t\to 0} \frac{1}{t}\Big(\phi\big(g \cdot \exp(t\alpha)\big) - \phi(g)\Big), \quad \Big(g \in GL(n,\mathbb{R})\Big).$$

We extend this to vector-valued functions by linearity:

$$D_\alpha \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix} := \begin{pmatrix} D_\alpha\phi_1 \\ \vdots \\ D_\alpha\phi_r \end{pmatrix}.$$

Then the algebra of differential operators with complex coefficients generated by the operators $D_\alpha, \alpha \in \mathfrak{gl}(n,\mathbb{R})$ is a realization of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} := \mathfrak{gl}(n,\mathbb{C})$. Its center, $Z(U(\mathfrak{g}))$ is isomorphic to a polynomial algebra in n generators. One choice of generators is given by the Casimir differential operators:

$$\left\{ \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_m=1}^{n} D_{i_1,i_2}\circ D_{i_2,i_3}\circ\cdots\circ D_{i_m,i_1} \;\middle|\; 1 \le m \le n \right\},$$

where for $1 \le i \le n,\ 1 \le j \le n$, we have $D_{i,j} := D_{E_{i,j}}$, and $E_{i,j} \in \mathfrak{gl}(n,\mathbb{R})$ is the $n \times n$ matrix with a 1 at the i, j entry and zeros everywhere else.

In order to have a well-defined action of the center $Z(U(\mathfrak{g}))$ of the universal enveloping algebra $U(\mathfrak{g})$ on the space of functions $GL(n,\mathbb{R})^+ \to \mathbb{C}^r$ satisfying the ρ-equivariance condition (12.3.13), it is necessary to check that the action of $Z(U(\mathfrak{g}))$ on the space of all smooth functions $GL(n,\mathbb{R})^+ \to \mathbb{C}^r$ preserves this space. This follows easily from the next proposition.

Proposition 12.3.15 (The elements of $Z(U(\mathfrak{g}))$ are invariant differential operators) *Let $\Phi : GL(n,\mathbb{R}) \to \mathbb{C}$ be a smooth function, and let D be an* *element of* $Z(U(\mathfrak{g}))$*. For* $g \in GL(n,\mathbb{R})$ *let* $\pi(g)$ *denote the action by right translation, given by* $(\pi(g)\,.\,\Phi)(h) := \Phi(h\cdot g)$*. Then*

$$D\,.\,\pi(g)\,.\,\Phi = \pi(g)\,.\,D\,.\,\Phi, \qquad (\forall g \in GL(n,\mathbb{R})).$$

Proof The identity to be proved may be put into the equivalent form

$$\pi(g)\circ D\circ \pi(g)^{-1} = D, \qquad (\forall g \in GL(n,\mathbb{R}),\ D \in Z(U(\mathfrak{g}))),$$

where $\circ$ denotes composition of operators. Clearly, it suffices to consider only the generators given in Definition 12.3.14. From the matrix identity

$$g\cdot \exp(t\alpha)\cdot g^{-1} = \exp(t\cdot(g\cdot\alpha\cdot g^{-1})), \qquad (\forall t \in \mathbb{R}, g \in GL(n,\mathbb{R}), \alpha \in \mathfrak{gl}(n,\mathbb{R})),$$

it follows that

$$\pi(g)\circ D_\alpha\circ \pi(g)^{-1} = D_{g\cdot\alpha\cdot g^{-1}}, \qquad (\forall g \in GL(n,\mathbb{R}),\ \alpha \in \mathfrak{gl}(n,\mathbb{R})).$$

Form an $n\times n$ matrix M of differential operators such that the i, j entry is $D_{i,j} := D_{E_{i,j}}$ where $E_{i,j} \in \mathfrak{gl}(n,\mathbb{R})$ is the matrix with a 1 at the i, j entry and zeros everywhere else. Observe that the degree m generator given in Definition 12.3.14 is simply the trace of the m-fold matrix product M^m. It follows easily from the linearity of the map $\alpha \to D_\alpha$ that the i, j entry of the matrix gMg^{-1} is $D_{g\cdot E_{i,j}\cdot g^{-1}}$. The invariance of each of the generators given in Definition 12.3.14 now follows easily from the invariance of the trace of a matrix under action by conjugation. □

Example The case $n = 2$ was considered previously in Section 5.1. The center of the universal enveloping algebra has two generators when $n = 2$, namely

$$D_{I_2} = D_{1,1}+D_{2,2}, \quad \text{and} \quad \Delta = D_{1,1}\circ D_{1,1}+D_{1,2}\circ D_{2,1}+D_{2,1}\circ D_{1,2}+D_{2,2}\circ D_{2,2}.$$

Here $D_{i,j} := D_{E_{i,j}}$, where $E_{i,j} \in \mathfrak{gl}(n,\mathbb{R})$ is the matrix with a 1 at the i, j entry and zeros everywhere else. Invariance of these differential operators was proved by a different method in Section 5.1. To check it via the method of Proposition 12.3.15, we consider the matrix of differential operators

$$M := \begin{pmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{pmatrix}.$$

Then $D_{1,1} + D_{2,2}$ is the trace of M, while Δ is the trace of

$$\begin{aligned} M^2 &= \begin{pmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{pmatrix}\cdot\begin{pmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{pmatrix} \\ &= \begin{pmatrix} D_{1,1}\circ D_{1,1} + D_{1,2}\circ D_{2,1} & D_{1,1}\circ D_{1,2} + D_{1,2}\circ D_{2,2} \\ D_{2,1}\circ D_{1,1} + D_{2,2}\circ D_{2,1} & D_{2,1}\circ D_{1,2} + D_{2,2}\circ D_{2,2} \end{pmatrix}. \end{aligned}$$

Cambridge University Press
978-1-107-00799-4 - Automorphic Representations and L-Functions for the General Linear Group, Volume II
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Excerpt
More information

Definition 12.3.16 (Action of $Z(U(\mathfrak{g}))$ on automorphic functions of weight ρ, level N and character χ) Let Φ be a vector valued automorphic function of weight ρ, level N and character χ as in Definition 12.3.9. Let $\widetilde{\Phi}$ denote the lift of Φ to $GL(n, \mathbb{R})^+$ as in Definition 12.3.12. For each $D \in Z(U(\mathfrak{g}))$, and each $z \in \mathfrak{h}^n$ define

$$(D \,.\, \Phi)(z) = (D\widetilde{\Phi})(z).$$

With this in hand, it is finally possible to give a generalization of Definition 3.5.7.

Definition 12.3.17 (Vector valued Maass form for $GL(n, \mathbb{R})$) Let $n \geq 2$, $r \geq 1$, $N \geq 1$ be integers. Fix a Dirichlet character $\chi \pmod N$, and an irreducible representation $\rho : SO(n, \mathbb{R}) \to GL(r, \mathbb{C})$. A vector valued Maass form of weight ρ and character χ for $\Gamma_0(N)$ is a smooth function $\Phi : \mathfrak{h}^n \to \mathbb{C}^r$ satisfying the following conditions

- $\left(\Phi\big|_{\rho}\gamma\right)(z) = \chi(d)\Phi(z), \qquad \text{for all } \gamma = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \in \Gamma_0(N), \;\; z \in \mathfrak{h};$
- the function Φ is an eigenfunction of every element of $Z(U(\mathfrak{g}))$, acting as in Definition 12.3.16;
- Φ is of moderate growth as in Definition 12.3.10;
- $\displaystyle\int_{\Gamma_0(N)\backslash\mathfrak{h}^n} ||\Phi(z)||^2 \, d^*z < \infty,$

where d^*z denotes the $GL(n, \mathbb{R})$-invariant measure given in [Goldfeld, 2006, 1.5], and $||\ ||$ denotes a positive definite norm on $\mathbb{C}^r$. (It is easy to see that any two such norms give the same condition here.)

A Maass form is said to be of level N, if it is a Maass form for $\Gamma_0(N)$, and it is not a Maass form for $\Gamma_0(M)$ with $M < N$.

Definition 12.3.18 (Factorization of $\mathfrak{h}^n$ corresponding to an integer $m < n$) For $n \geq 2$, let $\mathfrak{h}^n$ denote the generalized upper half plane associated to $GL(n, \mathbb{R})$ as in Definition 12.1.1. Let m be an integer with $1 \leq m < n$. We define a factorization of $\mathfrak{h}^n$ into two pieces which depend on m. As usual, I_r (for $r = 1, 2, 3, \dots$) denotes the $r \times r$ identity matrix. Define $X_m^n \subset \mathfrak{h}^n$ to be the set of all real valued matrices x' of the form

$$x' = \begin{pmatrix} I_m & X \\ 0 & I_{n-m} \end{pmatrix}, \qquad \left(\text{where } \; X = \begin{pmatrix} x_{1,m+1} & \dots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,m+1} & \dots & x_{m,n} \end{pmatrix}\right),$$

and let Y_m^n be the set of all real valued matrices y' of the form

$$y' = \begin{pmatrix} z'd & 0 \\ 0 & z'' \end{pmatrix}, \qquad \left(\text{where } \; z' \in \mathfrak{h}^m, \;\; z'' \in \mathfrak{h}^{n-m}, \;\; d = rI_m \text{ (with } r > 0)\right).$$