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Convexity

An Analytic Viewpoint

BARRY SIMON

California Institute of Technology

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9781107007314

First published 2011
Reprinted 2012

Printed and bound by CPI Group (UK) Ltd, Croydon CR0 4YY

A catalog record for this publication is available from the British Library

Library of Congress Cataloging in Publication data
Simon, Barry, 1946–
Convexity : an analytic viewpoint / Barry Simon.
p. cm.
Includes index.
ISBN 978-1-107-00731-4 (hardback)
1. Convex domains. 2. Mathematical analysis. I. Title.
QA639.5.S56 2011
516′.08 – dc22 2011008617

ISBN 978-1-107-00731-4 Hardback

Cambridge University Press
978-1-107-00731-4 - Convexity: An Analytic Viewpoint
Barry Simon
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Preface

Convexity of sets and functions are extremely simple notions to define, so it may be somewhat surprising the depth and breadth of ideas that these notions give rise to. It turns out that convexity is central to a vast number of applied areas, including Statistical Mechanics, Thermodynamics, Mathematical Economics, and Statistics, and that many inequalities, including Hölder's and Minkowski's inequalities, are related to convexity.

An introductory chapter (1) includes a study of regularity properties of convex functions, some inequalities (Hölder, Minkowski, and Jensen), the Hahn–Banach theorem as a statement about extending tangents to convex functions, and the introduction of two constructions that will play major roles later in this book: the Minkowski gauge of a convex set and the Legendre transform of a function.

The remainder of the book is roughly in four parts: convexity and topology on infinite-dimensional spaces (Chapters 2–5); Loewner's theorem (Chapters 6–7); extreme points of convex sets and related issues, including the Krein–Milman theorem and Choquet theory (Chapters 8–11); and a discussion of convexity and inequalities (Chapters 12–16).

The first part begins with a study of Orlicz spaces in Chapter 2, a notion that extends L^p. The most interesting new example is $L^1 \log L$ but the theory also illustrates parts of L^p theory. Chapter 3 introduces the notion of locally convex spaces and includes a discussion of L^p and H^p for $0 < p < 1$ to illustrate what can happen in nonlocally convex spaces. Among the issues discussed are uniqueness of topologies on $\mathbb{R}^n$ as a topological vector space, the fact that infinite-dimensional spaces are never locally compact, Kolmogorov's theorem that a topological vector space has a topology given by a norm if and only if 0 has a bounded convex neighborhood, Fréchet and barreled spaces. Chapter 4 deals with finding hyperplanes to slip between disjoint convex sets. It is an appealing geometric notion, mainly important for technical reasons. Chapter 5 discusses dual topologies and the Mackey–Arens theorem which describes all topologies in which Y is the dual of X where Y is a rich family of linear functionals on X. We also discuss Legendre transforms in great generality and prove Fenchel's theorem on when the double Legendre transform of a function is the function itself. Polar sets are a key technical tool.

The second part discusses Loewner's theorem and related ideas: when does a function, f, on (a, b) preserve matrix inequalities between matrices with eigenvalues in (a, b) and when is f convex applied to matrices. The answer is given by a deep theorem of Loewner that describes the set of such f: for the monotonicity question, they must be analytic on (a, b) and have an analytic continuation to all of $\mathbb{C}_+$ with $\operatorname{Im} f > 0$ if $\operatorname{Im} z > 0$! We describe the framework in Chapter 6 and the first proof of Loewner's theorem in Chapter 7 (there will be another proof in Chapter 9).

The third part focuses on geometric ideas, especially extreme points. Chapter 8 proves several basic results in this area, most notably the result that combines theorems of Minkowski and Carathéodory that any point $x \in K$, a compact convex subset of $\mathbb{R}^\nu$, is a convex combination of at most $\nu + 1$ extreme points, and the Krein–Milman theorem that a compact convex subset of a locally convex space is the closed convex hull of its extreme points. We begin the discussion of ergodic theory continued in the next chapter. Chapter 9 shows that if the set of extreme points is closed (often true, but it can even fail in the finite-dimensional case), then any point is an integral of extreme points. Applications include Bernstein's theorem on totally positive functions, Bochner's theorem, and a second proof of Loewner's theorem. There are several examples presented where the extreme points are dense rather than closed, showing the need for extending the representation theory to situations where the extreme points are not closed: that is the subject known as Choquet theory – existence is the topic of Chapter 10 and uniqueness Chapter 11. Uniqueness turns out to be associated to vector order, so that subject is partially discussed in Chapter 11.

The fourth and final part continues the discussion of convexity and inequalities. Chapter 12 discusses Hadamard's three-circle and three-line bounds in the theory of analytic functions, and applies it to the Riesz–Thorin and Stein interpolation theorems. Applications to Young's inequality and analyticity of L^p semigroups follow. Chapter 13 details a remarkable inequality of Prékopa about integrals of log concave functions. Applications include the Brunn–Minkowksi inequality for convex sets, the classical isoperimetric inequality, and an isoperimetric inequality for Dirichlet ground states. We also give a proof of the general Brunn–Minkowski inequality. Chapters 14 and 15 deal with two threads in rearrangement, both going back to work of Hardy, Littlewood, and Pólya. Chapter 14 focuses on the Brascamp–Lieb–Luttinger inequality, its proof including the study of Steiner symmetrization and applications including additional isoperimetric inequalities. Chapter 15 studies the issue of when $\int \varphi(|g(x)|)\, d\mu(x) \leq \int \varphi(|f(x)|)\, d\mu(x)$ for all even convex functions, φ, on $\mathbb{R}$. Along the way, we will prove Birkhoff's theorem identifying the extreme points in the set on $n \times n$ matrices, A, with $a_{ij} \geq 0$, and so that each row and each column sums to 1. Chapter 16 provides a variational principle for entropy based on Legendre transforms and uses it to prove a semicontinuity result of importance in spectral analysis.

While this book is extensive, there are numerous topics in convexity left out – some of them are indicated in the Notes (Chapter 17).

I'd like to thank Almut Burchard, Brian Davies, Leonid Golinskii, Helge Krüger, Elliott Lieb, Michael Loss, and especially Derek Robinson for useful comments about this book. As always, the love and support of my family, especially my wife Martha, was invaluable.

Barry Simon
Los Angeles