CAMBRIDGE

Cambridge University Press & Assessment
978-1-107-00625-6 — Ideal MHD
Jeffrey P. Freidberg
Excerpt
More Information

# 1

# Introduction

## 1.1 The role of MHD in fusion energy

Magnetohydrodynamics (MHD) is a fluid model that describes the macroscopic equilibrium and stability properties of a plasma. Actually, there are several versions of the MHD model. The most basic version is called "ideal MHD" and assumes that the plasma can be represented by a single fluid with infinite electrical conductivity and zero ion gyro radius. Other, more sophisticated versions are often referred to as "extended MHD" or "generalized MHD" and include finite resistivity, two-fluid effects, and kinetic effects (e.g. finite ion gyro radius, trapped particles, energetic particles, etc.). The present volume is focused on the ideal MHD model.

Most researchers agree that MHD equilibrium and stability are necessary requirements for a fusion reactor. If an equilibrium exists but is MHD unstable the result is almost always very undesirable. There can be a violent termination of the plasma known as a major disruption. If no disruption occurs, the result is likely to be a greatly enhanced thermal transport which is highly detrimental to fusion power balance. In order to avoid MHD instabilities it is necessary to limit the regimes of operation so that the plasma pressure and current are below critical values. However, these limiting values must still be high enough to meet the needs of producing fusion power. In fact it is fair to say that the main goal of ideal MHD is the discovery of stable, magnetically confined plasma configurations that have sufficiently high plasma pressure and current to satisfy the requirements of favorable power balance in a fusion reactor.

### *1.1.1 The plasma pressure in a fusion reactor*

To put the role of MHD in context with respect to fusion it is useful to quantify the value of plasma pressure required in a reactor. This is easily done by considering



© in this web service Cambridge University Press & Assessment www.cambridge.org

CAMBRIDGE
Cambridge University Press & Assessment
978-1-107-00625-6 — Ideal MHD
Jeffrey P. Freidberg
Excerpt
More Information



simple power balance in a deuterium–tritium (D–T) fusion plasma where the heating power produced by fusion alpha particles balances the thermal conduction losses due to classical collisions and plasma turbulence.[1] This balance must be achieved at an optimum temperature that maximizes fusion energy production. The resulting "ignited" plasma is self-sustaining, requiring no external heating sources. The power balance condition is given by

$$\begin{gathered}\text{alpha heating} = \text{thermal loss}\\ \frac{E_\alpha}{4} n^2 \langle \sigma v \rangle = \frac{3}{2}\frac{p}{\tau_E}\end{gathered} \tag{1.1}$$

where $E_\alpha = 3.5$ MeV, $n$ is the electron number density, $\langle \sigma v \rangle$ is the velocity averaged D–T fusion cross section, $p$ is the plasma pressure, and $\tau_E$ is the thermal conduction energy confinement time. For a plasma with equal temperatures $T_D = T_T = T_e \equiv T$, the plasma pressure is equal to $p = 2nT$, where $T$ is measured in units of energy. Some simple manipulations allow Eq. (1.1) to be rewritten in terms of one version of the Lawson (1957) parameter as follows:

$$p\tau_E = \frac{24}{E_\alpha}\frac{T^2}{\langle \sigma v \rangle} \tag{1.2}$$

For many years this fundamental requirement has divided fusion research into three main areas of study: heating, transport, and MHD. The reasoning for this division starts with the recognition that the function $T^2/\langle \sigma v \rangle$ has a minimum at approximately $T = 15$ keV. It is important to operate at this temperature or else $p$ and/or $\tau_E$ would have to raised, both of which lead to increased costs. It is the job of the heating community to provide ways to heat the plasma to about 15 keV.

At this temperature ignition requires

$$(p\tau_E)_{\text{min}} \approx 8 \text{ atm-sec} \tag{1.3}$$

Learning how to produce a plasma with a sufficiently long $\tau_E$ is the job of the transport community. Learning how to produce plasmas with a sufficiently large $p$ is the job of the MHD community. For many years these three areas of research were reasonably separated. As fusion research has progressed, longer duration, high-performance plasmas have been produced and these three areas have started to overlap. The reason is that plasma–wall interactions have become increasingly important and have a large, simultaneous impact on heating, transport, and MHD. For the moment it is, nonetheless, still useful to think of the three separate plasma requirements for an ignited plasma.

[1] Readers unfamiliar with fusion reactor power balance should refer to the Further reading at the end of the chapter.

© in this web service Cambridge University Press & Assessment www.cambridge.org

CAMBRIDGE
Cambridge University Press & Assessment
978-1-107-00625-6 — Ideal MHD
Jeffrey P. Freidberg
Excerpt
More Information



One might think on the basis of Eq. (1.3) that it might be possible to make tradeoffs between $p$ and $\tau_E$ in order to reach ignition in as easy a way as possible. In practice there is not much room for tradeoffs. The reason is that if one wants to construct a standard base-load reactor with a power output of 1 GWe as economically possible, this actually requires a specific value of $p$. The reasoning behind this conclusion is based on (1) the intuition that "most economical" translates into smallest size and (2) the smallest size is set by the maximum neutron flux passing through the first wall. The maximum allowable neutron wall loading as set by material limitations is typically assumed to be $P_W \approx 4\,\text{MW/m}^2$. The condition that the neutron flux not exceed the wall loading limit in a toroidal reactor is given by

$$\begin{gathered}\text{fusion neutron flux} = \text{wall loading} \\ \frac{E_n}{16} p^2 \frac{\langle \sigma v \rangle}{T^2} \left(2\pi^2 R_0 a^2\right) = P_W \left(4\pi^2 R_0 a\right)\end{gathered} \tag{1.4}$$

Here, $E_n = 14.1$ MeV, $R_0$ is the major radius of the torus, and $a$ is the minor radius. Solving for $p$ yields

$$p = \left(32 \frac{T^2}{\langle \sigma v \rangle} \frac{P_W}{E_n a}\right)^{1/2} \tag{1.5}$$

The minor radius of the plasma appearing in Eq. (1.5) can be accurately approximated by assuming that most of the electric power is produced by the fusion neutrons with a conversion efficiency $\eta \approx 0.4$. Thus, Eq. (1.4) can be rewritten as

$$\begin{gathered}\text{electric power} = \eta(\text{neutron power}) \\ P_E = \eta P_n = \eta P_W \left(4\pi^2 R_0 a\right)\end{gathered} \tag{1.6}$$

Now, the minor radius $a$ can be rewritten in terms of the dimensionless inverse aspect ratio $a/R_0$

$$a = \left(\frac{1}{4\pi^2} \frac{a}{R_0} \frac{P_E}{\eta P_W}\right)^{1/2} \tag{1.7}$$

Typically $R_0/a \sim 3$. The exact value is not too critical since it enters the value of the pressure as a fourth root. For the parameters under consideration one finds $a \approx 2.3$ m, which when substituted into Eq. (1.5) leads to

$$p \approx 7 \text{ atm} \tag{1.8}$$

The conclusion is that a fusion plasma must have a pressure of about 7 atm and a corresponding energy confinement time equal to 1.1 sec. In general there is some, but not a lot, of flexibility in these values.

© in this web service Cambridge University Press & Assessment www.cambridge.org

CAMBRIDGE
Cambridge University Press & Assessment
978-1-107-00625-6 — Ideal MHD
Jeffrey P. Freidberg
Excerpt
More Information



### 1.1.2 The dimensionless pressure, β

The analysis of MHD is almost always carried out in terms of a dimensionless pressure denoted by $\beta$. There are various detailed definitions in the literature, the most important of which are discussed in the text. All definitions involve the ratio of plasma pressure to applied magnetic pressure:

$$\beta \equiv \frac{p}{B^2/2\mu_0} \tag{1.9}$$

In configurations with a large toroidal magnetic field and an aspect ratio $R_0/a \sim 3$, the corresponding reactors typically require $\beta \sim 5-10\%$, values that have been already achieved experimentally. In tighter aspect ratio devices, higher stable $\beta$ values are attainable, but often the pressure is not higher because, for engineering and geometric reasons, the magnetic field is smaller. Other concepts do not rely on a large toroidal magnetic field, which is an important engineering advantage. As a result their required and achieved MHD $\beta$ values are higher. However, such configurations typically have poorer MHD stability behavior leading to enhanced thermal transport. Almost all discussions of MHD in the literature involve $\beta$, but readers should stay alert to the fact that it is pressure that is the critical parameter for a fusion reactor.

### 1.1.3 A variety of fusion concepts

What is the best magnetic geometry for a fusion reactor from the point of view of MHD? Over the years many ideas have been tried. A list is given below:

Belt pinch
Cusp
Elmo bumpy torus
Field reversed configuration
Force-free pinch
Heliac
High $\beta$ stellarator
Levitated dipole
Mirror
Octopole
Perhapsatron
Plasma focus
Reversed field pinch
Screw pinch
Spherical tokamak
Spheromak
Stellarator
Stuffed caulked cusp
Tandem mirror
Theta pinch
Tokamak
Tormac
Z-pinch
Z-pinch – hard-core

Clearly there has not been a shortage of imagination in inventing new concepts. Of this long list two concepts have risen to the top, largely because of superior overall plasma physics performance. These are the tokamak and the stellarator. It should be noted that while these configurations have the best plasma physics performance,

© in this web service Cambridge University Press & Assessment www.cambridge.org

CAMBRIDGE

Cambridge University Press & Assessment
978-1-107-00625-6 — Ideal MHD
Jeffrey P. Freidberg
Excerpt
More Information



they may not be the optimized choice from an engineering point of view. Both of these concepts have a large toroidal magnetic field which adds to the cost and complexity of a fusion reactor. Still, unless other concepts can overcome the plasma physics challenges their more desirable engineering features cannot be utilized. So far, while progress has been made, they have not yet been able to overcome these challenges, thereby explaining why tokamaks and stellarators remain at the top of the list.

### *1.1.4 Structure of the textbook*

The basic structure of the textbook is straightforward. The discussion begins with a description of the ideal MHD model and some of its general properties. This is followed by a discussion of MHD equilibrium in simple and general geometries. The last main topic discussed involves MHD stability.

There are many examples presented, although the bulk of the actual applications involve tokamaks and stellarators. There is also a substantial discussion of the reversed field pinch, a concept that is not as yet quite as advanced as tokamaks and stellarators in terms of performance. Still, it does hold some promise and its relatively simple geometric properties make it an ideal example to help understand MHD equilibrium and stability.

The overall purposes of the textbook are to provide both a qualitative and quantitative understanding of ideal MHD theory as applied to magnetic fusion. The specific goals are to discover concepts capable of achieving MHD stable, high-pressure, fusion-grade plasmas.

## 1.2 Units

The basic units used throughout the textbook are the usual SI units. The one exception is temperature, which always appears in conjunction with Boltzmann's constant, $k$. This constant is always absorbed into the temperature which then has the units of energy: $kT \to T$.

In the course of the text a number of relations are derived in terms of practical units as defined below:

| | | |
|---|---|---|
| Number density | $n$ | $10^{20}$ m$^{-3}$ |
| Temperature | $T$ | keV |
| Magnetic field | $B$ | T (tesla) |
| Current | $I$ | MA (megamperes) |
| Minor radius | $a$ | m |
| Major radius | $R_0$ | m |

© in this web service Cambridge University Press & Assessment www.cambridge.org

CAMBRIDGE

Cambridge University Press & Assessment
978-1-107-00625-6 — Ideal MHD
Jeffrey P. Freidberg
Excerpt
More Information



## References

Lawson, J. D. (1957), *Proceedings of the Physical Society* **B70**, 6.

## Further reading

### *The history of fusion*

Dean, Stephen O. (2013). *Search for the Ultimate Energy Source*. New York: Springer Press.

Fowler, T. Kenneth (1997). *The Fusion Quest*. Baltimore, MD: The John Hopkins University Press.

### *Power balance in a fusion reactor*

Dolan, T. J. (1982). *Fusion Research*. New York: Pergamon Press.

Freidberg, Jeffrey (2007). *Plasma Physics and Fusion Energy*. Cambridge: Cambridge University Press.

Kikuchi, Mitsuru, Karl Lackner, and Minh Quang Tran, editors (2012). *Fusion Physics*. Vienna: International Atomic Energy Agency.

Stacey, Weston M. (2005). *Fusion Plasma Physics*. Weinhein, Germany: Wiley-VCH.

Wesson, John (2011). *Tokamaks*, 4th edn. Oxford: Oxford University Press.

© in this web service Cambridge University Press & Assessment www.cambridge.org

CAMBRIDGE

Cambridge University Press & Assessment
978-1-107-00625-6 — Ideal MHD
Jeffrey P. Freidberg
Excerpt
More Information

# 2

# The ideal MHD model

## 2.1 Introduction

The goal of Chapter 2 is to provide a physical understanding of the ideal MHD model. Included in the discussion are (1) a basic description of the model, (2) a derivation starting from a more fundamental kinetic model, and, most importantly, (3) an examination of its range of validity.

In particular, it is shown that ideal MHD is the simplest fluid model that describes the macroscopic equilibrium and stability properties of a plasma. The claim of "simplest" is justified by a discussion of the large number of important plasma phenomena *not* covered by the model. However, in spite of its simplicity it is still a difficult model to solve analytically or even computationally because of the geometrical complexities associated with the two and three dimensionality of the configurations of fusion interest.

The derivation of the MHD model follows from the standard procedure of starting with a more fundamental and inclusive kinetic description of the plasma which describes the behavior of the electron and ion distribution functions. The mass, momentum, and energy moments of the kinetic equations are then evaluated. By introducing the characteristic length and time scales of ideal MHD, and making several corresponding ordering approximations, one is then able to close the system. The end result is the set of ideal MHD fluid equations.

The validity of the model is then assessed by examining the ordering assumptions used for closure to see whether or not they are consistent with the actual properties of fusion plasmas. This is a crucial step since ideal MHD is widely used in the design and interpretation of fusion experiments and one must be sure to understand the limits on the validity of the model. The assessment shows that while the basic derivation of MHD is straightforward there are several hidden surprises and subtleties.

Questions arise for two reasons. First, one of the basic assumptions used in the derivation, i.e., that the plasma is collision dominated, is *never* satisfied in plasmas



© in this web service Cambridge University Press & Assessment www.cambridge.org

CAMBRIDGE

Cambridge University Press & Assessment
978-1-107-00625-6 — Ideal MHD
Jeffrey P. Freidberg
Excerpt
More Information



of fusion interest. Even so, there is overwhelming empirical evidence that MHD provides an accurate description of macroscopic plasma behavior. This apparent good fortune is not a lucky coincidence but the consequence of some subtle physics; namely, those parts of the MHD model that are not valid because of violation of the collision dominated assumption are not directly involved in many if not most phenomena of interest. In other words, the model is only incorrect when it is unimportant. An attempt is made to clarify these issues in Chapter 9 by the introduction of several more sophisticated, low-collisionality plasma models whose regimes of validity are more closely aligned with actual experimental operating conditions. These models are more difficult to solve mathematically. However, several general equilibrium and stability comparison theorems are derived in Chapter 10 that help explain why ideal MHD works as well as it does.

The second subtle MHD issue concerns the following. Ideal MHD is an asymptotic model in the sense that specific length and time scales must be assumed for the derivation to be valid. In addition certain naturally appearing dimensionless parameters involving the MHD length and time scales must be ordered as small, medium, or large in order to close the system. For instance, high collisionality is represented by one such parameter. The issue here is that the multiple criteria defining the regime of validity arise from the need to simultaneously satisfy each assumption used in the derivation. However, a certain subset of phenomena described by the model requires only a corresponding subset of criteria to be satisfied, and consequently can have a much wider range of validity. One important example is MHD equilibrium. This important and useful information is discussed as the analysis progresses.

With these subtleties in mind attention is now focused on providing an in-depth description of the ideal MHD model.

## 2.2 Description of the model

The ideal MHD model provides a single-fluid description of long-wavelength, low-frequency, macroscopic plasma behavior. To put the model in perspective, it is perhaps useful to first discuss those plasma phenomena *not* described by ideal MHD.

Regarding physics in general, it has been pointed out that the three major discoveries of modern physics during the last two centuries, namely:

- Maxwell's equations with wave propagation
- relativity
- quantum mechanics

are each eliminated in the derivation of MHD.

© in this web service Cambridge University Press & Assessment www.cambridge.org

CAMBRIDGE
Cambridge University Press & Assessment
978-1-107-00625-6 — Ideal MHD
Jeffrey P. Freidberg
Excerpt
More Information



Within the narrower confines of plasma physics itself, there are a variety of phenomena important in fusion plasmas. Among them are:

- radiation
- RF heating and current drive
- resonant particle effects
- micro instabilities
- classical and anomalous transport
- plasma–wall interactions
- resistive instabilities
- $\alpha$-particle behavior.

Similarly, none of these phenomena is adequately described by ideal MHD.

Although the apparent lack of physical content is humbling, the one crucial phenomenon simply but accurately described by the model is the effect of magnetic geometry on the macroscopic equilibrium and stability of fusion plasmas. Specifically, ideal MHD answers such basic questions as: How does a given magnetic geometry provide forces to hold a plasma in equilibrium? Why are certain magnetic geometries more stable against macroscopic disturbances than others? Why do fusion configurations have such technologically undesirable shapes as a torus or a toroidal-helix?

One should be aware that in spite of the simplicity implied by its limited physical content, the ideal MHD model is still too difficult to solve in most geometries of interest. This will become evident as the text progresses by noting the many sophisticated expansions required to obtain analytic insight into the MHD behavior of various magnetic configurations. Attempts to solve similar problems using more comprehensive kinetic models are extremely difficult, even numerically, in realistic two- and three-dimensional geometries.

With this perspective the ideal MHD model is given by

$$
\begin{aligned}
&\text{Mass:} && \frac{\partial \rho}{\partial \mathrm{t}} + \nabla \cdot (\rho \mathbf{v}) = 0 \\
&\text{Momentum:} && \rho \frac{d\mathbf{v}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p \\
&\text{Energy:} && \frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0 \\
&\text{Ohm's law:} && \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \\
&\text{Maxwell:} && \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
& && \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \\
& && \nabla \cdot \mathbf{B} = 0
\end{aligned}
\tag{2.1}
$$

© in this web service Cambridge University Press & Assessment www.cambridge.org

CAMBRIDGE
Cambridge University Press & Assessment
978-1-107-00625-6 — Ideal MHD
Jeffrey P. Freidberg
Excerpt
More Information



In these equations, the electromagnetic variables are the electric field **E**, the magnetic field **B**, and the current density **J**. The fluid variables are the mass density $\rho$, the fluid velocity **v**, and the pressure $p$. Also, $\gamma = 5/3$ is the ratio of specific heats and $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ is the convective derivative.

Observe that in ideal MHD the electromagnetic behavior is governed by the low-frequency, pre-Maxwell equations. The MHD fluid equations describe the time evolution of mass, momentum, and energy.

The mass equation implies that the total number of plasma particles is conserved; phenomena such as ionization, recombination, charge exchange, and unfortunately fuel depletion by fusion reactions, are negligible to a high order of accuracy on the MHD time scale.

The basic physics of the momentum equation corresponds to that of a fluid with three interacting forces: the pressure gradient force $\nabla p$, the magnetic force $\mathbf{J} \times \mathbf{B}$, and the inertial force $\rho d\mathbf{v}/dt$. In static equilibrium it is the $\mathbf{J} \times \mathbf{B}$ force that balances the $\nabla p$ force, thereby confining the plasma. Dynamically, one must examine the stability of any such equilibrium to determine whether or not the plasma remains in place.

The energy equation expresses an adiabatic evolution characterized by a ratio of specific heats, $\gamma = 5/3$. The remaining relation is Ohm's law, which implies that in a reference frame moving with plasma the electric field is zero; that is, the plasma is a perfect conductor. It is the perfect conductivity assumption of Ohm's law that gives rise to the name "ideal" MHD.

As stated previously, the conditions for validity of the ideal MHD model imply that the phenomena of interest correspond to certain length and time scales. For macroscopic behavior the characteristic length scale is that of the overall plasma dimension. Denoting this dimension by $a$, then typically, for present day high-performance experiments, $a \sim 1$ m. The characteristic speed with which MHD phenomena occur is the thermal velocity of the plasma ions: $V_{Ti} = (2T_i/m_i)^{1/2}$, where $T_i$ is the ion temperature and $m_i$ is the ion mass. This gives rise to a characteristic MHD time $\tau_M \equiv a/V_{Ti}$. For $m_i$ equivalent to deuterium and $T_i = 3$ keV then $\tau_M \sim 2$ μsec. The MHD length and time scales are compared with those of other basic plasma physics phenomena in Tables 2.1 and 2.2. In computing these values it has been assumed that $a = 1$ m, $T_e = T_i = 3$ keV, $B = 5$ T, $n = 10^{20}$ m$^{-3}$ (particle number density) and $m_i$ equivalent to deuterium. Also the Coulomb logarithm has been set to $\ln \Lambda = 19$.

The richness of plasma physics is clearly evidenced by the large number and wide range of length and time scales. Among these, ideal MHD lies midway between a variety of high-frequency microscopic phenomena and low-frequency collisional transport phenomena. This is the regime of macroscopic equilibrium and stability.

© in this web service Cambridge University Press & Assessment www.cambridge.org