THE THEORY OF FUSION SYSTEMS

Fusion systems are a recent development in finite group theory and sit at the intersection of algebra and topology. This book is the first to deal comprehensively with this new and expanding field, taking the reader from the basics of the theory right to the state of the art.

Three motivational chapters, indicating the interaction of fusion and fusion systems in group theory, representation theory, and topology are followed by six chapters that explore the theory of fusion systems themselves. Starting with the basic definitions, the topics covered include: weakly normal and normal subsystems; morphisms and quotients; saturation theorems; results about control of fusion; and the local theory of fusion systems. At the end, there is also a discussion of exotic fusion systems.

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An Algebraic Approach

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Preface

It is difficult to pinpoint the origins of the theory of fusion systems: it could be argued that they stretch back to Burnside and Frobenius, with arguments about the fusion of p-elements of finite groups. Another viewpoint is that it really started with the theorems on fusion in finite groups, such as Alperin's fusion theorem, or Grün's theorems.

We will take as the starting point the important paper of Solomon [Sol74], which proves that, for a Sylow 2-subgroup P of $\mathrm{Spin}_7(3)$, there is a particular pattern of the fusion of involutions in P that, while not internally inconsistent, is not consistent with living inside a finite group. This is the first instance where the fusion of p-elements looks fine on its own, but is incompatible with coming from a finite group.

Unpublished work of Puig during the 1990s and even before (some of which is collected in [Pui06]), together with work of Alperin–Broué [AB79], is the basis for constructing a fusion system for a p-block of a finite group. It was with Puig's work where the axiomatic foundations of fusion systems started, and where some of the fundamental notions begin. It cannot be overestimated how much the current theory of fusion systems owes to Puig, both in originating the definition and related notions, and in furthering the theory.

Various results that could be considered part of local finite group theory (the study of p-subgroups, normalizers, conjugacy, and so on) were extended to p-blocks of finite groups during the 1990s and early part of the twenty-first century, but at the time were not viewed as taking place in the more general setting of fusion systems. With this theory now becoming more popular, more and more results are being cast in this language, and extended to this area.

The internal theory of fusion systems, starting from these foundations, has developed rapidly, and in many respects has mimicked the theory of finite groups, with normal subsystems, quotients, the generalized Fitting subsystem, composition series, soluble fusion systems, and so on. However, there is also a topological aspect to this theory.

Along with the representation theory, topology has played an important role in the development of the theory: Benson [Ben98a] constructed a topological space that should be the 2-completed classifying space of a finite group whose fusion pattern matched that which Solomon considered. Since such a group does not exist, this space can be thought of as the shadow cast by an invisible group. Benson predicted that this topological space is but one facet of a general theory, a prediction that was confirmed with the development of p-local finite groups.

Although we will not cover the topic of p-local finite groups here (we only meet the definition in Chapter 9), they can be thought of as some data describing a p-completed classifying space of a fusion system. In the case where the fusion system arises from a finite group, the corresponding p-local finite group describes the normal p-completed classifying space.

In this direction, we have Oliver's proof [Oli04] [Oli06] of the Martino–Priddy conjecture [MP96], which states that two finite groups have homotopy equivalent p-completed classifying spaces if and only if the fusion systems are isomorphic. The topological considerations have fuelled development in the algebraic aspects of fusion systems and vice versa, and the two viewpoints are somewhat intertwined. Having said that, we will not deal with the topological theory here beyond that which is given in Part I, and concentrate on the more algebraic aspects.

As this is a young subject, still in development, the foundations of the theory have not yet been solidified; indeed, there is some debate as to the correct *definition* of a fusion system! (It should be noted that the definitions are all equivalent, and so the choice is only apparent.) The definition of a 'normal' subsystem is also under discussion, and which definition is used often indicates the intended applications of the theory. Here we have made a choice based upon the evidence available now; this might change as time goes on. Since group theorists, representation theorists, and topologists all converge on this area, there are several different conventions and styles, as well as approaches.

The first three chapters are preliminary in nature, and deal with group theory, representation theory, and topology. The first chapter is essentially a run-through of the theory of fusion in finite groups, giving for example the p-complement theorems of Frobenius, Burnside, and Glauberman–Thompson. The second chapter introduces the representation theory aspects, and in particular develops the block theory needed to construct the fusion system associated to a p-block of a finite group. The third chapter develops the topological methods used in the theory, but since the main thrust of this work is the algebraic theory of fusion systems, we necessarily skip over many of the details in this chapter.

The remaining six chapters deal with the theory of fusion systems. The fourth chapter starts by defining fusion systems and in particular saturated fusion systems, then constructing local subsystems, proving Alperin's fusion theorem, and introducing strongly closed subgroups. In Chapter 5, we start looking at the normal and quotient structure of fusion systems, introducing morphisms, quotients, weakly normal and characteristic subsystems, the centre, and so on. The sixth chapter deals with methods used to prove saturation, and introduces weakly normal maps as well.

Chapter 7 deals with topics around control of fusion, with analogues of the Glauberman–Thompson normal p-complement theorem, Glauberman's ZJ-theorem, and the two normal subgroups $\mathrm{O}^p(G)$ and $\mathrm{O}^{p'}(G)$, the former of which is the object of the theory of transfer. After proving the existence of a certain kind of biset associated to any saturated fusion system in Section 7.6, we use the biset to develop the transfer for a fusion system.

Chapter 8 focuses on work of Aschbacher which attempts to translate some aspects of local finite group theory into the domain of fusion systems. We prove here that, for constrained fusion systems, there is a one-to-one correspondence between the normal subsystems of the fusion system and the normal subgroups of the associated model. Other highlights include a description of the generalized Fitting subsystem of a fusion system, and the proof of L-balance for fusion systems, which is considerably easier than the proof of the corresponding theorem for finite groups.

The final chapter consists of questions about exotic fusion systems (i.e., fusion systems that do not come from groups), with a few details on some of the known exotic fusion systems, theorems on which exotic fusion systems do not come from *blocks* of finite groups, and Oliver's conjecture relating modular representation theory of p-groups to the existence and uniqueness of centric linking systems. A solution to this conjecture would remove the requirement of the classification of the finite simple groups for the proof of the Martino–Priddy conjecture.

The choice of definitions and conventions has been influenced by the background of the author: as I am a group theorist and group representation theorist, the conventions here will be the standard group theory conventions, rather than topology conventions. In particular, homomorphisms will be composed from left to right. The only chapter where this will be relaxed is Chapter 3, the topological chapter; the reason for this is that to keep left-to-right notation would go against every other topology book in existence, and require writing functors on the right, something that I, even as a group theorist, cannot bring myself to do.

It remains for me to thank various people, most notably Adam Glesser for reading much of this work and for being a sounding board for various ideas, mathematical, pedagogical and notational. Thank you to George Raptis for reading Chapter 3, and explaining some of the topological ideas to an algebraist, making the exposition in that chapter considerably clearer. Proof reading and valuable comments were given by (in alphabetical order): Tobias Barthel, Michael Collins, Radha Kessar, Bob Oliver, Oscar Randal-Williams, George Raptis, Raphaël Rouquier, Jason Semeraro and Matt Towers. Any errors that remain in this work are, of course, my own.

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