Part I

Motivation

1

Fusion in finite groups

The fusion of elements of prime power order in a finite group is the source of many deep theorems in finite group theory. In this chapter we will briefly survey this area, and use this theory to introduce the notion of a fusion system of a finite group.

As this is the first chapter, we introduce some basic notation and terminology that we will use throughout the text. If G is a finite group and g is an element of G, we denote by c_g the conjugation map $c_g : G \to G$ given by $c_g : x \mapsto x^g = g^{-1}xg$. If H is a subgroup of G, then by c_H we mean the natural map $\mathrm{N}_G(H) \to \mathrm{Aut}(H)$ sending $g \in \mathrm{N}_G(H)$ to c_g. If K is another subgroup of G, the *automizer* of H (or rather, $\mathrm{N}_H(K)$) in K, denoted $\mathrm{Aut}_H(K)$, is the image of H under c_K, and is naturally isomorphic to $\mathrm{N}_H(K)/\mathrm{C}_H(K)$, so we will often identify $\mathrm{Aut}_H(K)$ and $\mathrm{N}_H(K)/\mathrm{C}_H(K)$. Write $\mathrm{Inn}(K) = \mathrm{Aut}_K(K)$, and by $\mathrm{Out}(K)$ and $\mathrm{Out}_H(K)$ we mean $\mathrm{Aut}(K)/\mathrm{Inn}(K)$ and $\mathrm{Aut}_H(K)\,\mathrm{Inn}(K)/\mathrm{Inn}(K)$ respectively.

If $\phi : H \to L$ is some isomorphism, then there is an induced map $\mathrm{Aut}(H) \to \mathrm{Aut}(L)$ such that, for $\psi \in \mathrm{Aut}(H)$,

$$\psi \mapsto \phi^{-1}\psi\phi.$$

If ψ is an element of $\mathrm{Aut}(H)$, we denote its image under this map as ψ^ϕ, and we will normally simply use ϕ to describe this induced map; no confusion should arise because the domains are different. If confusion could arise however, or we want to emphasize that it is this map we are considering, we will denote it by c_ϕ, since it is an analogue of the conjugation map c_g.

If x and y are elements of G, then x and y are H*-conjugate* if there is some h in H such that $x^h = y$, and we extend this definition to subgroups in the obvious way.

If π is a set of primes, then π' denotes all primes not in π. For any finite group G, $\mathrm{O}_\pi(G)$ denotes the largest normal π-subgroup, and $\mathrm{O}^\pi(G)$ denotes the smallest normal subgroup whose quotient is a π-group. As usual, if $\pi = p$ we simply write $\mathrm{O}_p(G)$ and $\mathrm{O}^p(G)$ respectively. (A standard fact that we will use often is that $\mathrm{O}^\pi(G)$ contains all elements of π'-order in G.) The *exponent* (denoted $\exp(G)$) is the lowest common multiple of all orders of all elements in the group.

A *p-local subgroup* is a subgroup of G of the form $\mathrm{N}_G(Q)$, for Q a non-trivial p-subgroup of G. Finally, the set of Sylow p-subgroups of a finite group G is denoted by $\mathrm{Syl}_p(G)$. We remind the reader that homomorphisms act on the right.

1.1 Control of fusion

We begin with a famous theorem.

Theorem 1.1 (Burnside) *Let G be a finite group, and let P be a Sylow p-subgroup of G. Suppose that P is abelian. Let x and y be two elements in a Sylow p-subgroup P of G. If x and y are G-conjugate then they are $\mathrm{N}_G(P)$-conjugate.*

Proof Let x and y be elements of P, and suppose that there is some $g \in G$ such that $x^g = y$, via $c_g : x \mapsto y$. We have that

$$P^g \leq \mathrm{C}_G(x)^g = \mathrm{C}_G(x^g) = \mathrm{C}_G(y),$$

and so both P and P^g are Sylow p-subgroups of $\mathrm{C}_G(y)$. Thus there exists $h \in \mathrm{C}_G(y)$ such that $P^{gh} = P$. Therefore $gh \in \mathrm{N}_G(P)$ and we have

$$x^{gh} = (x^g)^h = x^g,$$

and so $c_{gh} = c_g$ on x, as required. □

This theorem is a statement about the fusion of P-conjugacy classes in G.

Definition 1.2 Let G be a finite group, let H and K be subgroups of G with $H \leq K$, and let x and y be elements of H.

(i) If x and y are not conjugate in H, then x and y are *fused* in K if they are conjugate by an element of K. Similarly, two subgroups or two conjugacy classes of H are fused in K if they satisfy the obvious condition.

(ii) The subgroup K *controls weak fusion in* H *with respect to* G if, whenever x and y are fused in G, they are fused in K. (This is equivalent to the fusion of conjugacy classes.)

(iii) The subgroup K *controls* G*-fusion in* H if, whenever two subgroups A and B are conjugate via a conjugation map $c_g : A \to B$ for some $g \in G$, then there is some $k \in K$ such that c_g and c_k agree on A. (This is stronger than simply requiring any two subgroups conjugate in G to be conjugate in K.)

In the literature, control of weak fusion is often called 'control of fusion', and control of G-fusion is often called 'control of strong fusion'. However, when we get to fusion systems, control of weak fusion will be much less important than control of G-fusion.

It is easy to see that if K controls G-fusion in H, then K controls weak fusion in H with respect to G. The next example proves that the converse is not true.

Example 1.3 Let K be the group $\mathrm{GL}_3(2)$, which acts naturally on a 3-dimensional vector space V over $\mathbb{F}_2$. The order of $\mathrm{GL}_3(2)$ is 168, and so there are elements of orders 3 and 7 in K. In fact, we can find a subgroup of K of order 21. More specifically, let

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The element x has order 7, y has order 3, and y normalizes $\langle x \rangle$, so that $\langle x, y \rangle$ has order 21.

The vector space V is simply an elementary abelian group of order 8, and it is not difficult to see that if P is an elementary abelian group of order 8 then $\mathrm{Aut}(P) = \mathrm{GL}_3(2)$. Hence the subgroup $\langle x, y \rangle$ of K becomes a subgroup of $\mathrm{Aut}(P)$ of order 21. Let G denote the semidirect product of P by this subgroup of $\mathrm{Aut}(P)$, a (soluble) group of order 168; we will identify x and y with their counterparts in $\mathrm{Aut}(P)$, and in G. The element x of order 7 acts non-trivially on P, and so must permute the seven *involutions* – i.e., elements of order 2 – transitively. In particular, the subgroup $H = \langle P, x \rangle$ also has the property that all involutions of P are H-conjugate, and so H controls weak fusion in P with respect to G.

However, it is not difficult to prove that H does not control G-fusion in P, since there are subgroups of P of order 4 that are G-conjugate but not H-conjugate (and we leave it as an exercise to find such subgroups).

Thus a subgroup H controlling weak fusion in P with respect to G does not necessarily control G-fusion in P.

Given these definitions, Theorem 1.1 has the following restatement.

Theorem 1.4 *Let G be a finite group, and let P be a Sylow p-subgroup of G. If P is abelian, then the normalizer $\mathrm{N}_G(P)$ controls weak fusion in P with respect to G.*

In the proof of Theorem 1.1, if we replace x and y by subsets A and B of P, then we get that if $c_g : A \to B$ is a map in G then there is some $h \in \mathrm{N}_G(P)$ such that $c_g = c_h$ on A. In other words, we get the following theorem.

Theorem 1.5 *Let G be a finite group, and let P be a Sylow p-subgroup of G. If P is abelian, then the normalizer $\mathrm{N}_G(P)$ controls G-fusion in P.*

What we are saying is that any fusion inside a Sylow p-subgroup P of a finite group must take place inside its normalizer, at least if P is abelian. In general, this is not true.

Example 1.6 Let G be the group $\mathrm{GL}_3(2)$, the simple group of order 168. This group has a dihedral Sylow 2-subgroup P, generated by the two matrices

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that x and y are both involutions. In $\mathrm{GL}_3(2)$, all of the twenty-one involutions are conjugate, but this is not true in $\mathrm{N}_G(P)$, since we claim that $\mathrm{N}_G(P) = P$. To see this, let

$$z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the central involution in P. Notice that, since z has twenty-one conjugates in G, $\mathrm{C}_G(z)$ has order 8, so that $\mathrm{C}_G(z) = P$. Hence $\mathrm{C}_G(P)$ is a 2-group. Since $\mathrm{Aut}(P)$ is a 2-group (as P is dihedral), and $\mathrm{Aut}_G(P) = \mathrm{N}_G(P)/\,\mathrm{C}_G(P)$, we see that $\mathrm{N}_G(P)$ is also a 2-group; thus $\mathrm{N}_G(P) = P$. In fact, there is no 2-local subgroup H for which x and y are H-conjugate. However, all is not lost; let $Q_1 = \langle x, z\rangle$, and let $N_1 = \mathrm{N}_G(Q_1)$. The subgroup N_1 is isomorphic with the symmetric group on four letters, and

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so inside here Q_1 has the property that all of its non-identity elements are conjugate in the overgroup N_1; therefore x and z are conjugate in N_1.

Similarly, write $Q_2 = \langle y, z\rangle$ and $N_2 = \mathrm{N}_G(Q_2)$. The same statements apply, and so y and z are conjugate inside N_2. Thus x and y are conjugate, via z, inside 2-local subgroups.

This idea of fusion of p-elements not being controlled by a single subgroup, but two elements being conjugate 'in stages' by a collection of subgroups is important, and is the basis of Alperin's fusion theorem, which we shall see in Section 1.3.

The notions of fusion and control of fusion (particularly the stronger 'control of G-fusion'), are interesting for us, and we will explore the fusion and control of fusion in Sylow p-subgroups of finite groups, and more abstractly with the notion of fusion systems. For a group, we give the definition of a fusion system now.

Definition 1.7 Let G be a finite group and let P be a Sylow p-subgroup of G. The *fusion system* of G on P is the category $\trianglelefteq (G)$, whose objects are all subgroups of P and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(A, B) = \mathrm{Hom}_G(A, B),$$

the set of all (not necessarily surjective) maps $A \to B$ induced by conjugation by elements of G. The composition of morphisms is composition of maps.

This definition is meant to capture the notion of fusion of p-elements and p-subgroups in the group G. We give an example of a fusion system now.

Example 1.8 Let G be the group $\mathrm{GL}_3(2)$, considered in Example 1.6, and let P be the Sylow 2-subgroup given there, with the elements x, y and z as given. The subgroup P is isomorphic with D_8, so $\mathcal{F}_P(P)$ is simply all of the conjugation actions given by elements of P. For example, we have the (not surjective) map $\phi : \langle x\rangle \mapsto \langle x, z\rangle$ sending x to xz; this is realized by conjugation by y.

Consider the fusion system $\mathcal{F}_P(G)$, which contains $\mathcal{F}_P(P)$. We will simply describe the bijective maps in $\mathcal{F}_P(G)$, since all injective maps in $\mathrm{Hom}_G(A, B)$ are bijections followed by inclusions. There are bijections $\langle g\rangle \to \langle h\rangle$, where g and h are involutions. The two elements of order 4 are conjugate in P, so there is a map $\langle xy\rangle \to \langle xy\rangle$ sending xy to

$(xy)^3$. Finally, there are maps involving the Klein four-subgroups. Let $Q_1 = \langle x, z\rangle$ and $Q_2 = \langle y, z\rangle$, as before.

We first consider the maps in $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q_1, Q_1) = \mathrm{Aut}_{\mathcal{F}_P(G)}(Q_1)$. Since $\mathrm{N}_G(Q_1)$ is the symmetric group S_4, and $\mathrm{C}_G(Q_1) = Q_1$, we must have that $\mathrm{Aut}_{\mathcal{F}_G(P)}(Q_1) = \mathrm{Aut}_G(Q_1)$ is isomorphic with S_3, and so is the full automorphism group. (Similarly, $\mathrm{Aut}_{\mathcal{F}_P(G)}(Q_2) = \mathrm{Aut}(Q_2)$.) If ϕ is any map $Q_1 \to Q_2$ in $\mathcal{F}_P(G)$, then by composing with a suitably chosen automorphism of Q_2, we get all possible isomorphisms $Q_1 \to Q_2$. This would include the map ϕ where $\phi : x \mapsto y$ and $\phi : z \mapsto z$; then x and y would be conjugate in $\mathrm{C}_G(z) = \mathrm{N}_G(P) = P$, and this is not true. Therefore there are no maps between Q_1 and Q_2. (One may prove this more easily using the fact that Q_1 and Q_2 stabilize different maximal flags, but the above argument is more in keeping with the rest of the book.)

This shows that, although all of the non-identity elements in Q_1 are conjugate to all non-identity elements in Q_2 in $\mathcal{F}_P(G)$, the subgroups Q_1 and Q_2 are not isomorphic in $\mathcal{F}_P(G)$. This is why we take all *subgroups* of P in the fusion system, rather than merely all elements.

The fusion system is meant to capture the concept of control of fusion, and indeed it does.

Proposition 1.9 *Let G be a finite group and let P be a Sylow p-subgroup of G. Let H be a subgroup of G containing P. The subgroup H controls G-fusion in P if and only if $\mathcal{F}_P(G) = \mathcal{F}_P(H)$.*

Proof This is essentially a restatement of the definition of control of G-fusion, and which maps $\phi : A \to B$ lie in the fusion system. The details are left to the reader. □

We have the following corollary of this proposition, our first result about fusion systems proper.

Corollary 1.10 *Let G be a finite group and let P be a Sylow p-subgroup of G. If P is abelian, then*

$$\mathcal{F}_P(G) = \mathcal{F}_P(\mathrm{N}_G(P)).$$

1.2 Normal p-complements

One of the first applications of fusion of finite groups was to the question of whether a group has a normal p-complement.

Definition 1.11 A finite group G has a *normal p-complement*, or is said to be *p-nilpotent*, if $\mathrm{O}_{p'}(G) = \mathrm{O}^p(G)$, i.e., $G = H \rtimes P$, where $H = \mathrm{O}_{p'}(G)$ and P is a Sylow p-subgroup of G.

The first results on the question of whether a finite group has a normal p-complement are from Burnside and Frobenius. Burnside's theorem is generally proved as an application of transfer, which we shall meet briefly in Chapter 7 (but see, for example, [Asc00, Section 37], [Gor80, Section 7.3], or [Ros78, Chapter 10], and also Section 7.7).

Frobenius's normal p-complement theorem is a set of three conditions, each equivalent to the existence of a normal p-complement. Modern proofs of this theorem use, along with the transfer, some machinery from the theory of fusion in finite groups, like Grün's first theorem or Alperin's fusion theorem. We will state this normal p-complement theorem but not prove it until Section 1.4.

Theorem 1.12 (Frobenius's normal p-complement theorem) *Let G be a finite group, and let P be a Sylow p-subgroup of G. The following are equivalent:*

(i) *G possesses a normal p-complement;*
(ii) $\mathcal{F}_P(G) = \mathcal{F}_P(P)$*;*
(iii) *every p-local subgroup of G possesses a normal p-complement;*
(iv) *for every p-subgroup Q of G,* $\mathrm{Aut}_G(Q)$ *is a p-group.*

This is not, of course, exactly what Frobenius proved, but instead of $\mathcal{F}_P(G) = \mathcal{F}_P(P)$ there was a statement about conjugacy in the Sylow p-subgroup, which is easily equivalent.

From this result, we will deduce Burnside's normal p-complement theorem, which is a sufficient, but not necessary, condition to having a p-complement.

Theorem 1.13 (Burnside's normal p-complement theorem) *Let G be a finite group, and let P be a Sylow p-subgroup of G. If $P \leq \mathrm{Z}(\mathrm{N}_G(P))$ then G possesses a normal p-complement.*

Proof Since $P \leq \mathrm{Z}(\mathrm{N}_G(P))$, we must have that P is abelian. Therefore, $\mathcal{F}_P(G) = \mathcal{F}_P(\mathrm{N}_G(P))$ by Corollary 1.10. Furthermore, since P is central in $\mathrm{N}_G(P)$, we see that $\mathcal{F}_P(\mathrm{N}_G(P)) = \mathcal{F}_P(P)$, and so by Frobenius's normal p-complement theorem, G possesses a normal p-complement, as claimed. □

We can quickly derive a result of Cayley from Frobenius's normal p-complement theorem as well, proving that no simple group has a cyclic Sylow 2-subgroup.

Corollary 1.14 (Cayley) *Let G be a finite group of even order, and let P be a Sylow 2-subgroup of G. If P is cyclic, then G has a normal 2-complement.*

Proof Notice that, if Q is any cyclic 2-group of order 2^m, then $|\operatorname{Aut}(Q)|$ is itself a 2-group. (It is the size of the set

$$\{x \mid 0 < x < 2^m,\ x \text{ is prime to } 2^m\},$$

which has size 2^{m-1}.) Thus $\operatorname{Aut}_G(Q)$ is a 2-group for all subgroups Q of G, since Q is cyclic. Hence by Frobenius's normal p-complement theorem, G possesses a normal 2-complement, as claimed. (Alternatively, since $\mathrm{C}_G(P) = \mathrm{N}_G(P)$, one may use Burnside's normal p-complement theorem.) □

Example 1.15 We return to our familiar example, where $G = \mathrm{GL}_3(2)$ and P is the Sylow 2-subgroup considered above. Since $\mathcal{F}_P(G)$ is not $\mathcal{F}_P(P)$, we should have that $\operatorname{Aut}_{\mathcal{F}_P(G)}(Q)$ is not a 2-group, for some $Q \leq P$. As we saw, the automizers in G of Q_1 and Q_2, the Klein four subgroups of P, have order 6, confirming Frobenius's theorem in this case.

While Frobenius's normal p-complement theorem was a breakthrough, Thompson's normal p-complement theorem was a significant refinement. The original theorem of Thompson [Tho64] proved that, for odd primes, G possesses a normal p-complement if two particular subgroups possess normal p-complements. (Note that Thompson proved an earlier normal p-complement theorem in [Tho60].) Glauberman [Gla68a] refined this further, proving that, for odd primes, G possesses a normal p-complement if *one* particular p-local subgroup possesses a normal p-complement! Both Thompson's and Glauberman's results use the Thompson subgroup, which we will define now.

Definition 1.16 Let P be a finite p-group, and let $\mathscr{A}$ denote the set of all elementary abelian subgroups of P of maximal order. The *Thompson subgroup*, $J(P)$, is defined to be the subgroup generated by all elements of $\mathscr{A}$.

There are several similar definitions of the Thompson subgroup in the literature, but this one will be fine for our purposes. We are now in a position to state the theorems; for a proof of the second theorem, see also [Gor80, Theorem 8.3.1].

Theorem 1.17 (Thompson [Tho64]) *Let G be a finite group, and let P be a Sylow p-subgroup of G, where p is an odd prime. The group G has a normal p-complement if and only if $\mathrm{N}_G(J(P))$ and $\mathrm{C}_G(\mathrm{Z}(P))$ have normal p-complements.*

Theorem 1.18 (Glauberman–Thompson [Gla68a]) *Let G be a finite group, and let P be a Sylow p-subgroup of G, where p is an odd prime. The group G has a normal p-complement if and only if $\mathrm{N}_G(\mathrm{Z}(J(P)))$ has a normal p-complement.*

Note that both of these theorems were originally proved using different versions of the Thompson subgroup (and, indeed, different from each other as well) and some modifications need to be made in order for the original proofs to be valid for our version of $J(P)$.

It may seem very surprising that a single p-local subgroup controls whether the whole group possesses a normal p-complement, but this is indeed the case. This theorem tells us that, with $N = \mathrm{N}_G(\mathrm{Z}(J(P)))$, if $\mathcal{F}_P(N) = \mathcal{F}_P(P)$, then $\mathcal{F}_P(N) = \mathcal{F}_P(G)$. Thus one way of looking at this theorem is that it gives a sufficient condition for N to control G-fusion in P.

In fact, this happens much more often. Glauberman's ZJ-theorem is a sufficient condition for this subgroup N given above to control G-fusion in P. It holds for odd primes, and for every group that does not involve a particular group $Qd(p)$, as a subquotient. Let p be a prime, and let $Q = C_p \times C_p$: this can be thought of as a 2-dimensional vector space, and so $\mathrm{SL}_2(p)$ acts on this group in a natural way. Define $Qd(p)$ to be the semidirect product of Q and $\mathrm{SL}_2(p)$.

Example 1.19 In the case where $p = 2$, the group $Qd(p)$ has a normal elementary abelian subgroup of order 4, and is the semidirect product of this group and $\mathrm{SL}_2(2) = S_3$. Hence, $Qd(2) = S_4$, the symmetric group on four letters.

Proposition 1.20 *Let G be the group $Qd(p)$, and let P be a Sylow p-subgroup of G. Then $\mathcal{F}_P(G) \neq \mathcal{F}_P(N)$, where $N = \mathrm{N}_G(\mathrm{Z}(J(P)))$.*

Proof The Sylow p-subgroup of $\mathrm{SL}_2(p)$ is cyclic, of order p, and so P is non-abelian of order p^3. Since P is a split extension of $C_p \times C_p$ by C_p, it has exponent p by Exercise 1.3. As every subgroup of index p is elementary abelian (and of maximal order), the Thompson subgroup of