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978-1-107-00529-7 - Lectures on Profinite Topics in Group Theory

Benjamin Klopsch, Nikolay Nikolov and Christopher Voll

Table of Contents

[More information](#)

# Contents

Preface	page ix
Editor's introduction	1
<b>I An introduction to compact <math>p</math>-adic Lie groups</b>	<b>7</b>
<i>by Benjamin Klopsch</i>	
1 Introduction . . . . .	7
2 From finite $p$ -groups to compact $p$ -adic Lie groups . . . . .	10
2.1 Nilpotent groups . . . . .	10
2.2 Finite $p$ -groups . . . . .	11
2.3 Lie rings . . . . .	12
2.4 Applying Lie methods to groups . . . . .	13
2.5 Absolute values . . . . .	15
2.6 $p$ -adic numbers . . . . .	16
2.7 $p$ -adic integers . . . . .	17
2.8 Preview: $p$ -adic analytic pro- $p$ groups . . . . .	18
3 Basic notions and facts from point-set topology . . . . .	19
4 First series of exercises . . . . .	21
5 Powerful groups, profinite groups and pro- $p$ groups . . . . .	25
5.1 Powerful finite $p$ -groups . . . . .	25
5.2 Profinite groups as Galois groups . . . . .	28
5.3 Profinite groups as inverse limits . . . . .	29
5.4 Profinite groups as profinite completions . . . . .	30
5.5 Profinite groups as topological groups . . . . .	31
5.6 Pro- $p$ groups . . . . .	32
5.7 Powerful pro- $p$ groups . . . . .	33
5.8 Pro- $p$ groups of finite rank – summary of characterisations . . . . .	34
6 Second series of exercises . . . . .	35
7 Uniformly powerful pro- $p$ groups and $\mathbb{Z}_p$ -Lie lattices . . . . .	39
7.1 Uniformly powerful pro- $p$ groups . . . . .	39
7.2 Associated additive structure . . . . .	40
7.3 Associated Lie structure . . . . .	41

Cambridge University Press

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Table of Contents

[More information](#)

vi

Contents

7.4	The Hausdorff formula . . . . .	42
7.5	Applying the Hausdorff formula . . . . .	43
8	The group $\mathrm{GL}_d(\mathbb{Z}_p)$ , just-infinite pro- $p$ groups and the Lie correspondence for saturable pro- $p$ groups . . . . .	44
8.1	The group $\mathrm{GL}_d(\mathbb{Z}_p)$ – an example . . . . .	44
8.2	Just-infinite pro- $p$ groups . . . . .	46
8.3	Potent filtrations and saturable pro- $p$ groups . . . . .	47
8.4	Lie correspondence . . . . .	48
9	Third series of exercises . . . . .	49
10	Representations of compact $p$ -adic Lie groups . . . . .	53
10.1	Representation growth and Kirillov’s orbit method . . . . .	53
10.2	The orbit method for saturable pro- $p$ groups . . . . .	54
10.3	An application of the orbit method . . . . .	56
	<b>References for Chapter I</b>	<b>57</b>
<b>II</b>	<b>Strong approximation methods</b>	<b>63</b>
	<i>by Nikolay Nikolov</i>	
1	Introduction . . . . .	63
2	Algebraic groups . . . . .	64
2.1	The Zariski topology on $K^n$ . . . . .	64
2.2	Linear algebraic groups as closed subgroups of $\mathrm{GL}_n(K)$ . . . . .	66
2.3	Semisimple algebraic groups: the classification of simply connected algebraic groups over $K$ . . . . .	73
2.4	Reductive groups . . . . .	76
2.5	Chevalley groups . . . . .	77
3	Arithmetic groups and the congruence topology . . . . .	77
3.1	Rings of algebraic integers in number fields . . . . .	78
3.2	The congruence topology on $\mathrm{GL}_n(k)$ and $\mathrm{GL}_n(\mathcal{O})$ . . . . .	78
3.3	Arithmetic groups . . . . .	80
4	The strong approximation theorem . . . . .	82
4.1	An aside: Serre’s conjecture . . . . .	84
5	Lubotzky’s alternative . . . . .	85
6	Applications of Lubotzky’s alternative . . . . .	87
6.1	The finite simple groups of Lie type . . . . .	87
6.2	Refinements . . . . .	87
6.3	Normal subgroups of linear groups . . . . .	89
6.4	Representations, sieves and expanders . . . . .	89
7	The Nori–Weisfeiler theorem . . . . .	90
7.1	Unipotently generated subgroups of algebraic groups over finite fields . . . . .	92
8	Exercises . . . . .	93
	<b>References for Chapter II</b>	<b>95</b>

Cambridge University Press

978-1-107-00529-7 - Lectures on Profinite Topics in Group Theory

Benjamin Klopsch, Nikolay Nikolov and Christopher Voll

Table of Contents

[More information](#)

<i>Contents</i>	vii
<b>III A newcomer's guide to zeta functions of groups and rings</b>	<b>99</b>
<i>by Christopher Voll</i>	
1 Introduction . . . . .	99
1.1 Zeta functions of groups . . . . .	99
1.2 Zeta functions of rings . . . . .	101
1.3 Linearisation . . . . .	103
1.4 Organisation of the chapter . . . . .	104
2 Local and global zeta functions of groups and rings . . . . .	105
2.1 Rationality and variation with the prime . . . . .	106
2.2 Flag varieties and Coxeter groups . . . . .	108
2.3 Counting with $p$ -adic integrals . . . . .	110
2.4 Linear homogeneous diophantine equations . . . . .	114
2.5 Local functional equations . . . . .	116
2.6 A class of examples: 3-dimensional $p$ -adic anti-symmetric algebras . . . . .	125
2.7 Global zeta functions of groups and rings . . . . .	126
3 Variations on a theme . . . . .	127
3.1 Normal subgroups and ideals . . . . .	127
3.2 Representations . . . . .	129
3.3 Further variations . . . . .	137
4 Open problems and conjectures . . . . .	138
4.1 Subring and subgroup zeta functions . . . . .	138
4.2 Representation zeta functions . . . . .	139
5 Exercises . . . . .	140
<b>References for Chapter III</b>	<b>141</b>
<b>Index</b>	<b>145</b>