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978-1-107-00529-7 - Lectures on Profinite Topics in Group Theory

Benjamin Klopsch, Nikolay Nikolov and Christopher Voll

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# Lectures on Profinite Topics in Group Theory

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# Preface

In September 2007, the London Mathematical Society and the EPSRC sponsored a ‘short course for graduates’ in Oxford, under the heading ‘Asymptotic methods in infinite group theory’. This was organised by Dan Segal and consisted of three series of lectures. The present book is basically a record of these lectures, somewhat polished and expanded. It is intended to serve as an introduction, for beginning research students and for interested non-specialists, to some areas of current activity in algebra: the questions mostly originate in group theory but the methodology encompasses a wide range of mathematics, involving topology, algebraic geometry, number theory and combinatorics.