

ENCYCLOPEDIA OF SPECIAL FUNCTIONS:
THE ASKEY–BATEMAN PROJECT

Volume 2: Multivariable Special Functions

This is the second of three volumes that form the *Encyclopedia of Special Functions*, an extensive update of the Bateman Manuscript Project.

Volume 2 covers multivariable special functions. When the Bateman project appeared, study of multivariable special functions was in an early stage, but revolutionary developments began to be made in the 1980s and have continued ever since. World-renowned experts survey these over the course of 12 chapters, each containing an extensive bibliography. The reader encounters different perspectives on a wide range of topics, from Dunkl theory, to Macdonald theory, to the various deep generalizations of classical hypergeometric functions to the several variables case, including the elliptic level. Particular attention is paid to the close relation of the subject with Lie theory, geometry, mathematical physics and combinatorics.

TOM H. KOORNWINDER is Professor Emeritus at the University of Amsterdam. He is an expert in special functions, orthogonal polynomials and Lie theory. He introduced the five-parameter extension of the BC-type Macdonald polynomials, which are nowadays called Koornwinder polynomials. He was co-author of the chapter on orthogonal polynomials in the *Digital Library of Mathematical Functions*, and is involved in its revision.

JASPER V. STOKMAN is Professor of Mathematics at the University of Amsterdam. He is an expert in special functions, Lie theory and integrable systems. He introduced BC-type extensions of several families of classical orthogonal polynomials, and nonpolynomial generalizations of Koornwinder polynomials. He linked multivariable special functions to harmonic analysis on quantum groups and Hecke algebras, and to statistical mechanics and analytic number theory.

Cambridge University Press

978-1-107-00373-6 — Encyclopedia of Special Functions: The Askey-Bateman Project

Edited by Tom H. Koornwinder , Jasper V. Stokman

Frontmatter

[More Information](#)

ENCYCLOPEDIA OF SPECIAL FUNCTIONS: THE ASKEY–BATEMAN PROJECT

Volume 2: Multivariable Special Functions

Edited by

TOM H. KOORNWINDER

Universiteit van Amsterdam

JASPER V. STOKMAN

Universiteit van Amsterdam



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press

978-1-107-00373-6 — Encyclopedia of Special Functions: The Askey-Bateman Project

Edited by Tom H. Koornwinder, Jasper V. Stokman

Frontmatter

[More Information](#)

CAMBRIDGE UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781107003736

DOI: 10.1017/9780511777165

© Cambridge University Press 2021

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2021

Printed in the United Kingdom by TJ International Ltd, Padstow Cornwall

A catalogue record for this publication is available from the British Library.

ISBN – 3 Volume Set 978-1-108-88244-6 Hardback

ISBN – Volume 1 978-0-521-19742-7 Hardback

ISBN – Volume 2 978-1-107-00373-6 Hardback

ISBN – Volume 3 978-0-521-19039-8 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Contents

<i>List of Contributors</i>	<i>page</i> ix
<i>Preface</i>	xi
1 General Overview of Multivariable Special Functions	1
<i>T. H. Koornwinder and J. V. Stokman</i>	
1.1 Introduction	1
1.2 Multivariable Classical, Basic and Elliptic Hypergeometric Series	3
1.3 Multivariable (Bi)Orthogonal Polynomials and Functions	10
1.4 Multivariable (Bi)Orthogonal Polynomials and Functions, Some Details	11
<i>References</i>	14
2 Orthogonal Polynomials of Several Variables	19
<i>Yuan Xu</i>	
2.1 Introduction	19
2.2 General Properties of Orthogonal Polynomials of Several Variables	20
2.3 Orthogonal Polynomials of Two Variables	29
2.4 Spherical Harmonics	42
2.5 Classical Orthogonal Polynomials of Several Variables	47
2.6 Relation Between Orthogonal Polynomials on Classical Domains	56
2.7 Orthogonal Expansions and Summability	59
2.8 Discrete Orthogonal Polynomials of Several Variables	61
2.9 Other Orthogonal Polynomials of Several Variables	67
<i>References</i>	72
3 Appell and Lauricella Hypergeometric Functions	79
<i>K. Matsumoto</i>	
3.1 Introduction	79
3.2 Appell's Hypergeometric Series	80
3.3 Lauricella's Hypergeometric Series	80
3.4 Integral Representations	81
3.5 Systems of Hypergeometric Differential Equations	84
3.6 Local Solution Spaces	89
3.7 Transformation Formulas	90

3.8	Contiguity Relations	91
3.9	Monodromy Representations	92
3.10	Twisted Period Relations	95
3.11	The Schwarz Map for Lauricella's F_D	96
3.12	Reduction Formulas	97
	<i>References</i>	97
4	A-Hypergeometric Functions	101
	<i>N. Takayama</i>	
4.1	Introduction	101
4.2	A-Hypergeometric Equations	101
4.3	Combinatorics, Polytopes and Gröbner Basis	106
4.4	A-Hypergeometric Series	107
4.5	Hypergeometric Function of Type $E(k, n)$	113
4.6	Contiguity Relations	114
4.7	Properties of A-Hypergeometric Equations	115
4.8	A-Hypergeometric Polynomials and Statistics	118
	<i>References</i>	119
5	Hypergeometric and Basic Hypergeometric Series and Integrals Associated with Root Systems	122
	<i>M. J. Schlosser</i>	
5.1	Introduction	122
5.2	Some Identities for (Basic) Hypergeometric Series Associated with Root Systems	125
5.3	Hypergeometric and Basic Hypergeometric Integrals Associated with Root Systems	143
5.4	Basic Hypergeometric Series with Macdonald Polynomial Argument	146
5.5	Remarks on Applications	151
	<i>References</i>	152
6	Elliptic Hypergeometric Functions Associated with Root Systems	159
	<i>H. Rosengren and S. O. Warnaar</i>	
6.1	Introduction	159
6.2	Integrals	164
6.3	Series	169
6.4	Elliptic Macdonald–Koornwinder Theory	174
	<i>References</i>	183
7	Dunkl Operators and Related Special Functions	187
	<i>C. F. Dunkl</i>	
7.1	Introduction	187
7.2	Root Systems	188
7.3	Invariant Polynomials	192
7.4	Dunkl Operators	193

7.5	Harmonic Polynomials	200
7.6	The Intertwining Operator and the Dunkl Kernel	203
7.7	The Dunkl Transform	208
7.8	The Poisson Kernel	209
7.9	Harmonic Polynomials for \mathbb{R}^2	210
7.10	Nonsymmetric Jack Polynomials	212
	<i>References</i>	215
8	Jacobi Polynomials and Hypergeometric Functions Associated with Root Systems	217
	<i>G. J. Heckman and E. M. Opdam</i>	
8.1	The Gauss Hypergeometric Function	217
8.2	Root Systems	219
8.3	The Hypergeometric System	220
8.4	Jacobi Polynomials	224
8.5	The Calogero–Moser System	232
8.6	The Hypergeometric Function	236
8.7	Special Cases	248
	<i>References</i>	254
9	Macdonald–Koornwinder Polynomials	258
	<i>J. V. Stokman</i>	
9.1	Introduction	258
9.2	The Basic Representation of the Extended Affine Hecke Algebra	265
9.3	Monic Macdonald–Koornwinder Polynomials	274
9.4	Double Affine Hecke Algebras and Normalized Macdonald–Koornwinder Polynomials	289
9.5	Explicit Evaluation and Norm Formulas	300
A	Appendix	301
	<i>References</i>	309
10	Combinatorial Aspects of Macdonald and Related Polynomials	314
	<i>J. Haglund</i>	
10.1	Introduction	314
10.2	Basic Theory of Symmetric Functions	315
10.3	Analytic and Algebraic Properties of Macdonald Polynomials	323
10.4	The Combinatorics of the Space of Diagonal Harmonics	332
10.5	The Expansion of the Macdonald Polynomial into Monomials	341
10.6	Consequences of Theorem 10.5.3	345
10.7	Nonsymmetric Macdonald Polynomials	350
10.8	The Genesis of the q, t -Catalan Statistics	355
10.9	Other Directions	358
10.10	Recent Developments	359
	<i>References</i>	361

11 Knizhnik–Zamolodchikov-Type Equations, Selberg Integrals and Related Special Functions	368
<i>V. Tarasov and A. Varchenko</i>	
11.1 Introduction	368
11.2 Representation Theory	369
11.3 Rational KZ Equation and Gaudin Model	370
11.4 Hypergeometric Solutions of the Rational KZ and Dynamical Equations, and Bethe Ansatz	373
11.5 Trigonometric KZ Equation	377
11.6 Hypergeometric Solutions of the Trigonometric KZ Equation and Bethe Ansatz	379
11.7 Knizhnik–Zamolodchikov–Bernard Equation and Elliptic Hypergeometric Functions	381
11.8 qKZ Equation	385
11.9 Hypergeometric Solutions of the qKZ Equations and Bethe Ansatz	388
11.10 One-integration Examples	392
11.11 Selberg-Type Integrals	394
11.12 Further Development	396
<i>References</i>	397
12 $9j$-Coefficients and Higher	402
<i>J. Van der Jeugt</i>	
12.1 Introduction	402
12.2 Representations of the Lie Algebra $\mathfrak{su}(2)$	403
12.3 Clebsch–Gordan Coefficients and $3j$ -Coefficients	404
12.4 Racah Coefficients and $6j$ -Coefficients	408
12.5 The $9j$ -Coefficient	412
12.6 Beyond $9j$: Graphical Methods	416
<i>References</i>	417
<i>Index</i>	420

Contributors

- Charles F. Dunkl *Department of Mathematics, University of Virginia, P.O. Box 400137, Charlottesville, VA 22904-4137, USA*
- Jim Haglund *Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA*
- Gert J. Heckman *IMAPP, Radboud Universiteit, P.O. Box 9010, 6500 GL Nijmegen, Netherlands*
- Tom H. Koornwinder *Korteweg-de Vries Institute for Mathematics, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, Netherlands*
- Keiji Matsumoto *Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan*
- Eric M. Opdam *Korteweg-de Vries Institute for Mathematics, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, Netherlands*
- Hjalmar Rosengren *Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE-412 96 Göteborg, Sweden*
- Michael J. Schlosser *Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria*
- Jasper V. Stokman *Korteweg-de Vries Institute for Mathematics, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, Netherlands*
- Nobuki Takayama *Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan*
- Vitaly Tarasov *Department of Mathematical Sciences, Indiana University–Purdue University Indianapolis, Indianapolis, IN 46202-3216, USA, and St. Petersburg Branch of Steklov Mathematical Institute, Fontanka 27, St. Petersburg, 191023, Russia*
- Joris Van der Jeugt *Department of Applied Mathematics, Computer Science and Statistics, Faculty of Sciences, Ghent University, Krijgslaan 281-S9, 9000 Ghent, Belgium*
- Alexander Varchenko *Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA*
- S. Ole Warnaar *School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia*
- Yuan Xu *Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA*

Cambridge University Press

978-1-107-00373-6 — Encyclopedia of Special Functions: The Askey-Bateman Project

Edited by Tom H. Koornwinder , Jasper V. Stokman

Frontmatter

[More Information](#)

Preface

This is the second volume of the *Encyclopedia of Special Functions* of the Askey–Bateman project. It is devoted to multivariable special functions.

As was explained in the preface to volume 1, the *Encyclopedia of Special Functions* aims to realize a vision that the late Richard Askey had in the 1970s: to update the Bateman project, in particular the three volumes of *Higher Transcendental Functions*, according to present knowledge and state of the art. As for multivariable special functions, the Bateman project contained material on Appell hypergeometric functions (part of Chapter V) and orthogonal polynomials in several variables (Chapter XII). These two most classical parts of multivariable special functions are treated in the present volume in Chapters 3 and 2, respectively.

In the past 65 years, since the Bateman project appeared, multivariable orthogonal polynomials and special functions have seen several revolutionary developments which partially interacted with each other and which also were fed by new insights into one-variable theory (notably basic and elliptic hypergeometric functions, and Askey–Wilson polynomials). One development was the successive introduction of zonal polynomials, hypergeometric functions of matrix argument, Jack polynomials, Hall–Littlewood polynomials, Heckman–Opdam polynomials (Chapter 8), Macdonald polynomials and Koornwinder’s extension of Macdonald’s BC case (Chapter 9), and Rains’ elliptic generalization of the Koornwinder polynomials (Chapter 6). Dunkl’s simultaneous introduction of the Dunkl operator (Chapter 7) and, a little later, Cherednik’s double affine Hecke algebras (Chapter 9) gave important boosts to these theories. Macdonald theory was also in fruitful interaction with algebraic combinatorics (Chapter 10). Analysis on semisimple Lie groups (Chapter 8) and quantum groups was also an important inspiration.

A second line of development was the quest for multivariable analogues of hypergeometric functions (which should be deep enough that many one-variable formulas generalize). The Appell hypergeometric functions turned out to be special cases of the A -hypergeometric functions (Chapter 4), introduced by Gel’fand and coworkers. Work by, among others, Biedenharn and coworkers resulted in many classes of multivariable hypergeometric series with expansion coefficients patterned by root systems; see Chapter 5 for the classical and basic cases, and Chapter 6 for the elliptic case. A very different kind of hypergeometric function associated with root systems, generalizing the theory of spherical functions on noncompact Riemannian symmetric spaces, was developed by Heckman and Opdam (Chapter 8). Yet another source of

multivariable hypergeometric functions, also closely connected with conformal field theory, comes from solving Knizhnik–Zamolodchikov-type equations (Chapter 11).

Wigner and Racah coefficients in the representation theory of $SU(n)$ and their application to quantum mechanics were the historical context from which the theory on multivariable hypergeometric series described in Chapter 5 arose. The case of $SU(2)$ is described in Chapter 12. Here the $9j$ -coefficients give rise to still mysterious orthogonal polynomials in two variables.

A more detailed survey of the chapters and their interconnections is given in the introductory Chapter 1.

We hope that the volume will help the reader to oversee the global landscape of multivariable special functions and their applications, and will serve as a useful guide to the extensive literature. We are very grateful to the authors of the chapters for their contributions to this volume. The final editing of the individual chapters and the creation of the index to the volume was done by the first editor.

Tom H. Koornwinder and Jasper V. Stokman