ENCYCLOPEDIA OF SPECIAL FUNCTIONS: THE ASKEY–BATEMAN PROJECT

Volume 2: Multivariable Special Functions

This is the second of three volumes that form the *Encyclopedia of Special Functions*, an extensive update of the Bateman Manuscript Project.

Volume 2 covers multivariable special functions. When the Bateman project appeared, study of multivariable special functions was in an early stage, but revolutionary developments began to be made in the 1980s and have continued ever since. World-renowned experts survey these over the course of 12 chapters, each containing an extensive bibliography. The reader encounters different perspectives on a wide range of topics, from Dunkl theory, to Macdonald theory, to the various deep generalizations of classical hypergeometric functions to the several variables case, including the elliptic level. Particular attention is paid to the close relation of the subject with Lie theory, geometry, mathematical physics and combinatorics.

TOM H. KOORNWINDER is Professor Emeritus at the University of Amsterdam. He is an expert in special functions, orthogonal polynomials and Lie theory. He introduced the five-parameter extension of the BC-type Macdonald polynomials, which are nowadays called Koornwinder polynomials. He was co-author of the chapter on orthogonal polynomials in the *Digital Library of Mathematical Functions*, and is involved in its revision.

JASPER V. STOKMAN is Professor of Mathematics at the University of Amsterdam. He is an expert in special functions, Lie theory and integrable systems. He introduced BC-type extensions of several families of classical orthogonal polynomials, and nonpolynomial generalizations of Koornwinder polynomials. He linked multivariable special functions to harmonic analysis on quantum groups and Hecke algebras, and to statistical mechanics and analytic number theory.

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Preface

This is the second volume of the *Encyclopedia of Special Functions* of the Askey–Bateman project. It is devoted to multivariable special functions.

As was explained in the preface to volume 1, the *Encyclopedia of Special Functions* aims to realize a vision that the late Richard Askey had in the 1970s: to update the Bateman project, in particular the three volumes of *Higher Transcendental Functions*, according to present knowledge and state of the art. As for multivariable special functions, the Bateman project contained material on Appell hypergeometric functions (part of Chapter V) and orthogonal polynomials in several variables (Chapter XII). These two most classical parts of multivariable special functions are treated in the present volume in Chapters 3 and 2, respectively.

In the past 65 years, since the Bateman project appeared, multivariable orthogonal polynomials and special functions have seen several revolutionary developments which partially interacted with each other and which also were fed by new insights into one-variable theory (notably basic and elliptic hypergeometric functions, and Askey–Wilson polynomials). One development was the successive introduction of zonal polynomials, hypergeometric functions of matrix argument, Jack polynomials, Hall–Littlewood polynomials, Heckman–Opdam polynomials (Chapter 8), Macdonald polynomials and Koornwinder's extension of Macdonald's BC case (Chapter 9), and Rains' elliptic generalization of the Koornwinder polynomials (Chapter 6). Dunkl's simultaneous introduction of the Dunkl operator (Chapter 7) and, a little later, Cherednik's double affine Hecke algebras (Chapter 9) gave important boosts to these theories. Macdonald theory was also in fruitful interaction with algebraic combinatorics (Chapter 10). Analysis on semisimple Lie groups (Chapter 8) and quantum groups was also an important inspiration.

A second line of development was the quest for multivariable analogues of hypergeometric functions (which should be deep enough that many one-variable formulas generalize). The Appell hypergeometric functions turned out to be special cases of the *A*-hypergeometric functions (Chapter 4), introduced by Gel'fand and coworkers. Work by, among others, Biedenharn and coworkers resulted in many classes of multivariable hypergeometric series with expansion coefficients patterned by root systems; see Chapter 5 for the classical and basic cases, and Chapter 6 for the elliptic case. A very different kind of hypergeometric function associated with root systems, generalizing the theory of spherical functions on noncompact Riemannian symmetric spaces, was developed by Heckman and Opdam (Chapter 8). Yet another source of multivariable hypergeometric functions, also closely connected with conformal field theory, comes from solving Knizhnik–Zamolodchikov-type equations (Chapter 11).

Wigner and Racah coefficients in the representation theory of SU(n) and their application to quantum mechanics were the historical context from which the theory on multivariable hypergeometric series described in Chapter 5 arose. The case of SU(2) is described in Chapter 12. Here the 9j-coefficients give rise to still mysterious orthogonal polynomials in two variables.

A more detailed survey of the chapters and their interconnections is given in the introductory Chapter 1.

We hope that the volume will help the reader to oversee the global landscape of multivariable special functions and their applications, and will serve as a useful guide to the extensive literature. We are very grateful to the authors of the chapters for their contributions to this volume. The final editing of the individual chapters and the creation of the index to the volume was done by the first editor.

Tom H. Koornwinder and Jasper V. Stokman