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General Overview of Multivariable Special Functions

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1.1 Introduction

The theory of one-variable (ordinary) hypergeometric and basic hypergeometric series goes back to work of Euler, Gauss and Jacobi. The theory of elliptic hypergeometric series is of a much more recent vintage (Frenkel and Turaev, 1997). The three theories deal with the study of series $\sum_{k\geq 0} c_k$ with $f(k) := c_{k+1}/c_k$ a rational function in k (*hypergeometric* theory), a rational function in q^k (*basic hypergeometric* theory) or a doubly periodic meromorphic function in k (*elliptic hypergeometric* theory; see Gasper and Rahman, 2004, Ch. 11 for an overview).

Examples of elementary functions admitting hypergeometric and basic hypergeometric series representations are

$$(1-z)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} z^k, \qquad \frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k \tag{1.1.1}$$

for $|z| < 1$ and $\alpha, a \in \mathbb{C}$, with $(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1)$ for $k \in \mathbb{Z}_{\geq 0}$ the *shifted factorial* (or *Pochhammer symbol*), $(a;q)_k := (1-a)(1-qa)\cdots(1-q^{k-1}a)$ for $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ the *q-shifted factorial*. Here, and throughout the entire chapter, we assume for convenience that $0 < q < 1$. Note that the series in the second identity, with $a = q^\alpha$, tends to the series in the first identity as $q \uparrow 1$, at least formally, and that the identities (1.1.1) reduce to polynomial identities when $\alpha \in \mathbb{Z}_{\leq 0}$. Also note that the series in (1.1.1) are indeed hypergeometric and basic hypergeometric series, respectively, since $f(k) = \frac{k+\alpha}{k+1} z$ and $f(k) = \frac{1-q^k a}{1-q^{k+1}} z$ for the first and second series in (1.1.1). These are the well-known Newton (generalized) *binomial theorem* and its q-analogue (Gasper and Rahman, 2004, §1.3). They form, apart from the (q-)exponential series, the simplest nontrivial examples of an impressive scheme of hypergeometric and basic hypergeometric summation identities (Gasper and Rahman, 2004), with the members in the scheme related by limit transitions.

The summands of elliptic, basic and classical hypergeometric series are expressible in terms of products and quotients of elliptic, basic and classical shifted factorials. The basic and classical ones are the (q-)shifted factorials as defined in the previous paragraph. The *elliptic* (or *theta*) *shifted factorial* is given by $(z;q,p)_k := \prod_{i=0}^{k-1} \theta(zq^i;p)$ for $k \in \mathbb{Z}_{\geq 0}$ and $0 < p < 1$, with $\theta(z;p) := \prod_{i=0}^{\infty}(1-p^i z)(1-p^{i+1}/z)$ the *modified theta function*. These shifted factorials

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can be expressed as $\Gamma(q^k z)/\Gamma(z)$ (or, in the classical case, $\Gamma(z+k)/\Gamma(z)$) with $\Gamma(z)$ an appropriate analogue of the classical Gamma function. For the elliptic hypergeometric case this is Ruijsenaars' (1997) *elliptic Gamma function*

$$\prod_{i,j=0}^{\infty} \frac{1 - z^{-1} p^{i+1} q^{j+1}}{1 - z p^i q^j}, \qquad 0 < p, q < 1,$$

for the basic hypergeometric case the (modified) *q-Gamma function* $(z;q)_\infty^{-1}$ and for the classical hypergeometric case the classical *Gamma function*.

There is no "simple" elliptic analogue of (1.1.1). In fact, the first elliptic hypergeometric summation formula that was found (Frenkel and Turaev, 1997) generalizes the *top-level* terminating (basic) hypergeometric summation identity! This is a general pattern for the elliptic hypergeometric theory: the top levels of the (basic) hypergeometric theory admit elliptic versions, and there is little room for degenerations without falling outside the realm of elliptic hypergeometric series. Possibly this is one of the reasons for the late discovery of elliptic hypergeometric series.

Parallel to the theory of hypergeometric series there is a theory of hypergeometric integrals; see §1.2.3 and, in later chapters, §5.3 and §6.2. Such integrals can often be identified with hypergeometric series. But, certainly in the elliptic case, there are many instances where the hypergeometric integral is convergent while a possible corresponding hypergeometric series diverges (Rosengren, 2017, §2.10). Hypergeometric integrals naturally appear as coordinates of vector-valued solutions of Knizhnik–Zamolodchikov (*KZ*) and Knizhnik–Zamolodchikov–Bernard (*KZB*) equations and their *q*-analogues; see Chapter 11. The elliptic case, corresponding to solutions of *qKZB* equations, appeared in Felder et al. (1997, §7) (yet formally) and soon afterwards rigorously in Felder et al. (1999, §6).

This volume deals with multivariable generalizations of ordinary, basic and elliptic hypergeometric series and integrals. This includes various multivariable extensions of classical (bi)orthogonal polynomials and functions, which form an important subclass of hypergeometric series within the one-variable theory.

Various multivariable theories have emerged, each with its own characteristic features depending on the particular motivation for, and context behind, its multivariable extension. For instance, there are important multivariable theories motivated by special function theory itself (see Chapters 2–6), by representation theory and Lie theory (see Chapters 7–9 and 12), by combinatorics (see Chapter 10) and by theoretical physics (see Chapters 8–9 and 11–12).

In the remainder of this introductory chapter we give a short discussion of each of the types of multivariable special functions treated in this volume, and we highlight their interrelations and differences. In §1.2 we first discuss the multivariable series which may be seen as extensions of the three types of hypergeometric series. The different classes of multivariable extensions of classical (bi)orthogonal functions will be discussed in §1.3.

We hope that this short impression of the various classes of multivariable special functions and their interrelations helps the reader to oversee the chapters in this volume and how they are related.

1.2 Multivariable Classical, Basic and Elliptic Hypergeometric Series

1.2.1 Appell and Lauricella Hypergeometric Series

Gauss' hypergeometric series is given by

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \tag{1.2.1} $$

which absolutely converges for $|x| < 1$. One of the oldest generalizations of the Gauss hypergeometric series to several variables was given by Appell, who introduced the four *Appell hypergeometric series* in two variables (see Appell and Kampé de Fériet, 1926; Erdélyi, 1953, §5.7), denoted by F_1, F_2, F_3, F_4. For instance,

$$ F_2(a, b_1, b_2, c_1, c_2; x, y) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n m!\,n!} x^m y^n, \quad (x, y) \in \mathbb{C}^2,\ |x| + |y| < 1. \tag{1.2.2} $$

The series (1.2.2) are double series $\sum_{m,n=0}^{\infty} c_{m,n}/(m!\,n!)$ with $c_{m+1,n}/c_{m,n}$ and $c_{m,n+1}/c_{m,n}$ of the form $p_1(m,n)/r_1(m,n)$ and $p_2(m,n)/r_2(m,n)$ for suitable relative prime polynomials p_i and r_i in two variables ($i = 1, 2$). This extends the property characterizing hypergeometric series in one variable, and such series are therefore also called *hypergeometric*. The highest degree of the four polynomials p_1, r_1, p_2, r_2 is called the *order* of the hypergeometric series in two variables. The Appell hypergeometric series have order two. Horn classified all hypergeometric series of order two; see the list of 34 series in Erdélyi (1953, §5.7.1). Lauricella defined n-variable analogues F_A, F_B, F_C, F_D of F_2, F_3, F_4, F_1, respectively. The Appell and Lauricella hypergeometric series are discussed in Chapter 3.

Many properties and formulas for Gauss hypergeometric series generalize to Appell and Lauricella hypergeometric series, but, not surprisingly, one has to deal with interesting complications concerning, for instance, the integral representations, systems of partial differential equations and monodromy; see Chapter 3. Furthermore, solutions of the system of partial differential equations for these series form a much richer collection than in the case of the Gauss hypergeometric series, where all local solutions at regular singularities are expressed in terms of series of the same type. For instance, for F_2 six different types of series occur as local solutions, including some that are hypergeometric series of order higher than two, or even not hypergeometric series at all; see Olsson (1977). Gel'fand's A-hypergeometric functions (see Chapter 4) offer a fruitful point of view for the study of Appell and Lauricella hypergeometric series. This can also give inspiration for a study of q-analogues; see Noumi (1992), where also a connection is made with quantum groups. Gasper and Rahman (2004, Chapter 10) give an account of q-series in two or more variables.

Appell and Lauricella hypergeometric series have several interrelations with other special functions in several variables. The first example (which may have been the motivating example for Appell) is the biorthogonal polynomials on the simplex and ball; see Chapter 2. Further examples deal with Heckman–Opdam hypergeometric functions (see Chapter 8). In the case of root system A, these functions can be identified for certain degenerate parameter values with a special Lauricella F_D (or, in two variables, with Appell F_1); see Shimeno and Tamaoka (2015). A special case of Heckman–Opdam hypergeometric functions for root system BC_2 can be written as the sum of two Appell F_4 functions (see Beerends, 1992, Theorems 3.3 and 2.3). In the polynomial case one of the F_4 terms vanishes, so that special BC_2 Jacobi polynomials can be written as a terminating F_4; see Koornwinder and Sprinkhuizen-Kuyper (1978, (7.15)). Another special case of the Heckman–Opdam functions for BC_n, now for general n, can be expressed as n-variable analogues of *Kampé de Fériet hypergeometric series*, certain hypergeometric series in two variables of order three (Beerends, 1992, (5.1) and Theorem 5.4).

1.2.2 A-Hypergeometric Functions

The A-hypergeometric (or *GKZ* hypergeometric) functions were introduced by Gel'fand et al. (1989), but there have been analogous approaches before. In particular, Miller Jr. (1973) described a new approach to the *hypergeometric differential equation*

$$z(1-z)f''(z) + (c-(a+b+1)z)f'(z) - abf(z) = 0, \tag{1.2.3}$$

of which the Gauss hypergeometric series (1.2.1) is a solution. He observed that if the parameters a, b, c in (1.2.3) are replaced by $s\partial_s$, $u\partial_u$, $t\partial_t$, then the resulting system of PDEs

$$QF = 0, \quad s\partial_s F = aF, \quad u\partial_u F = bF, \quad t\partial_t F = cF \tag{1.2.4}$$

with

$$Q := z(1-z)\partial_{zz} + t\partial_{tz} - z(s\partial_{sz} + u\partial_{uz} + \partial_z) - su\partial_{su}$$

has a solution

$$F(s,u,t,z) = s^a u^b t^c \, {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}; z\right). \tag{1.2.5}$$

Miller defines the *dynamical symmetry algebra* $\mathcal{G}$ of Q as the set of all first-order PDEs L such that $QLf = 0$ whenever $Qf = 0$. It is a Lie algebra with a basis of operators acting on solutions of the form (1.2.5) (so-called *contiguity relations*). Then $\mathcal{G}$ is seen to be isomorphic to sl(4). Miller Jr. (1973) pointed out that a similar approach works for generalized hypergeometric series ${}_{r+1}F_r$ and for Appell and Lauricella hypergeometric series. This was elaborated on by him in several papers in 1972, 1973.

Kalnins et al. (1980) transformed systems like (1.2.4), in the case of Appell's and Horn's hypergeometric series in two variables, into so-called *canonical systems*. These systems coincide with special cases of the later-introduced A-hypergeometric systems (Gel'fand et al., 1989). M. Saito (1996, 2001) recognized the relevance of Kalnins et al. (1980) for the *GKZ* theory. He also worked with a symmetry algebra for operators Q which no longer requires that the operators in the algebra are first order.

A change of variables turns system (1.2.4) of PDEs into the following canonical (or A-hypergeometric) form:

$$(\partial_{xy} - \partial_{zw})f = 0, \quad (x\partial_x - y\partial_y)f = (1-c)f, \quad (x\partial_x + z\partial_z)f = -af, \quad (x\partial_x + w\partial_w)f = -bf, \tag{1.2.6}$$

with corresponding solution

$$f(x,y,z,w) = y^{c-1}z^{-a}w^{-b}\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};\frac{xy}{zw}\right); \tag{1.2.7}$$

see for instance Stienstra (2007, §2.6, §3.2.1). Note that the change of variables has transformed the second-order partial differential operator Q to $\partial_{xy} - \partial_{zw}$, which is essentially the four-dimensional Laplace operator. This makes it manifest that the dynamical symmetry algebra of Q is $\mathfrak{sl}(4)$; see also Dereziński and Majewski (2016).

The general A-hypergeometric system in n variables $x = (x_1,\dots,x_n)$ depends on a $d\times n$ matrix $A = (a_{ij}) = (a_1 \dots a_n)$ with integer column vectors $a_j \in \mathbb{Z}^d$ (from which the A in A-hypergeometric) such that the $\mathbb{Z}$-span of the a_j equals $\mathbb{Z}^d$. The A-hypergeometric system, depending on parameters $\beta_1,\dots,\beta_d$, is given by

$$\Bigl(\prod_{u_i>0}\partial_{x_i}^{u_i}\Bigr)f = \Bigl(\prod_{u_i<0}\partial_{x_i}^{-u_i}\Bigr)f \quad (u\in L\backslash\{0\}), \qquad \Bigl(\sum_{j=1}^{n} a_{ij}x_j\partial_{x_j}\Bigr)f = \beta_i f \quad (i=1,\dots,d) \tag{1.2.8}$$

with $L := \{u\in\mathbb{Z}^n \mid Au = 0\}$. It can be seen to have system (1.2.6) as the special case

$$A = \begin{pmatrix}1 & -1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}, \qquad \beta = (1-c,-a,-b)^t. \tag{1.2.9}$$

For $\nu\in\mathbb{C}^n$ such that $A\nu = \beta$, we have a formal solution of the (ν-independent) differential equations (1.2.8) given by the series

$$\sum_{u\in L}\prod_{j=1}^{n}\frac{x_j^{\nu_j+u_j}}{\Gamma(\nu_j+u_j+1)}, \tag{1.2.10}$$

called *A-hypergeometric series in Gamma function form*. With the choice (1.2.9) of A,β and with $\nu := (0,c-1,-a,-b)^t$, $u := k(1,1,-1,-1)^t$ ($k\in\mathbb{Z}$) the series (1.2.10) becomes

$$\sum_{k=-\infty}^{\infty}\frac{x_1^k x_2^{c-1+k}x_3^{-a-k}x_4^{-b-k}}{\Gamma(k+1)\Gamma(c+k)\Gamma(-a-k+1)\Gamma(-b-k+1)} = \frac{x_2^{c-1}x_3^{-a}x_4^{-b}}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\,k!}\left(\frac{x_1x_2}{x_3x_4}\right)^k,$$

which is (1.2.7) apart from the Gamma factors in the denominator in front of the summation.

Choices for A and ν in (1.2.10) can be made such that the resulting series involves ${}_{r+1}F_r(z)$ or an Appell or Lauricella hypergeometric series. For instance, for Appell's F_2 one can take

$$A = \begin{pmatrix}1&0&0&0&0&1&1\\0&1&0&0&0&1&0\\0&0&1&0&0&0&1\\0&0&0&1&0&-1&0\\0&0&0&0&1&0&-1\end{pmatrix}, \qquad \begin{aligned}\beta &= (-a,-b_1,-b_2,c_1-1,c_2-1)^t,\\ u &= m(-1,-1,0,1,0,1,0)+n(-1,0,-1,0,1,0,1)^t,\\ \nu &= (-a,-b_1,-b_2,c_1-1,c_2-1,0,0)^t.\end{aligned}$$

Then the first part of system (1.2.8) is generated by $\partial_1\partial_2 f = \partial_4\partial_6 f$, $\partial_1\partial_3 f = \partial_5\partial_7 f$, and (1.2.10) becomes

$$\sum_{m,n=-\infty}^{\infty} \frac{x_1^{-a-m-n} x_2^{-b_1-m} x_3^{-b_2-n} x_4^{c_1+m-1} x_5^{c_2+n-1} x_6^m x_7^n}{\Gamma(m+1)\Gamma(n+1)\Gamma(c_1+m)\Gamma(c_2+n)\Gamma(1-a-m-n)\Gamma(1-b_1-m)\Gamma(1-b_2-n)}$$
$$= \frac{x_1^{-a} x_2^{-b_1} x_3^{-b_2} x_4^{c_1-1} x_5^{c_2-1}}{\Gamma(c_1)\Gamma(c_2)\Gamma(1-a)\Gamma(1-b_1)\Gamma(1-b_2)} F_2\left(a, b_1, b_2, c_1, c_2; \frac{x_4x_6}{x_1x_2}, \frac{x_5x_7}{x_1x_3}\right).$$

The *GKZ* theory, of which Chapter 4 gives a survey, not only unifies the study of many classes of multivariable special functions, but also exploits methods from algebra, geometry, *D*-module theory and combinatorics, far beyond the methods used in classical approaches.

1.2.3 Classical, Basic and Elliptic Hypergeometric Series and Integrals Associated with Root Systems

Hypergeometric integrals of classical, basic and elliptic type are integrals with integrand expressed in terms of products and quotients of Gamma factors $\Gamma(ax)$ (in the classical case, $\Gamma(a+x)$), with $\Gamma(x)$ the Gamma function of the appropriate type. In the classical case integrands involving products of the form $(1-x)^a$ are also considered to be hypergeometric ($(1-x)^a$ is formally the $q \to 1$ limit of the quotient $(q^x;q)_\infty/(q^{a+x};q)_\infty$ of q-Gamma functions). The singular set of the integrand of a hypergeometric integral is a union of geometric (in the classical case, arithmetic) progressions. Hypergeometric series naturally arise as the sum of residues of the integrand over such pole progressions.

Multidimensional hypergeometric integrals typically arise in contexts involving representation theory of algebraic and Lie groups. For instance, in harmonic analysis on compact symmetric spaces, the zonal spherical functions give rise to a family of multivariable orthogonal polynomials with respect to a measure on a compact torus that is absolutely continuous with respect to the Haar measure. The associated weight function admits a natural factorization in terms of the root system underlying the symmetric space. Such multivariable integrals often admit generalizations beyond the representation-theoretic context. They provide the prototypical examples of *hypergeometric integrals associated with root systems*.

Let us focus now more closely on the structure of such integrals. Suppose R is an irreducible root system in $\mathbb{R}^n$, and fix a choice R^+ of positive roots. The co-weight lattice $P^\vee$ of R is the lattice in $\mathbb{R}^n$ dual to the $\mathbb{Z}$-span of R. For the classical root systems we take the usual realization of $R = R^+ \cup (-R^+)$ in $\mathbb{R}^n$ with respect to the standard orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$. Concretely, $R^+ = \{e_i - e_j\}_{1\le i<j\le n}$ for type A_{n-1}, $R^+ = \{e_i \pm e_j\}_{1\le i<j\le n}$ for type D_n, $R^+ = \{e_i \pm e_j\}_{1\le i<j\le n} \cup \{2e_i\}_{i=1}^n$ for type C_n and $R^+ = \{e_i \pm e_j\}_{1\le i<j\le n} \cup \{e_i, 2e_i\}_{i=1}^n$ for type BC_n.

Let $k_\alpha \in \mathbb{C}$ be parameters that depend only on the Weyl group orbit of the root $\alpha \in R$ (equivalently, k_α depends only on the root length $\|\alpha\|$ of the root $\alpha \in R$). The prototypical example of a classical hypergeometric integral associated with R is

$$\int_{A_R} w_{\boldsymbol{k}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \qquad w_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{\alpha\in R} \left(1 - e^{2\pi i(\alpha,\boldsymbol{x})}\right)^{k_\alpha} \tag{1.2.11}$$

with $A_R \subset \mathbb{R}^n$ a fundamental domain for the translation action of $P^\vee$ on $\mathbb{R}^n$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1 \cdots \mathrm{d}x_n$ and $\boldsymbol{k} := \{k_\alpha\}_{\alpha \in R}$ the collection of the parameters k_α (here the k_α should satisfy appropriate conditions to ensure convergence of the integral). Remarkably the integral (1.2.11) admits an explicit evaluation as a product of Gamma functions. The resulting identity is known as the *Macdonald constant term identity* (see Theorem 8.4.2(i)). It gives the volume of the orthogonality measure of root system generalizations of the Jacobi polynomials, also known nowadays as *Heckman–Opdam polynomials*; see Chapter 8 for a detailed discussion.

Of particular interest is the special case that the root system R is of type BC_n. In that case the Macdonald constant term identity reduces after the change of variables $z_j = \sin^2(\pi x_j)$ to the well-known *Selberg integral* (Selberg, 1944)

$$\int_{[0,1]^n} \prod_{i=1}^{n} z_i^{\alpha-1}(1-z_i)^{\beta-1} \prod_{1\le i<j\le n} |z_i - z_j|^{2\gamma}\, \mathrm{d}\boldsymbol{z} = \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(1+\gamma)}$$

with parameters $\alpha = k_{\epsilon_1} + k_{2\epsilon_1} + \frac{1}{2}, \beta = k_{2\epsilon_1} + \frac{1}{2}$ and $\gamma = k_{\epsilon_1-\epsilon_2}$, which in turn is a multidimensional generalization of the beta integral. There are many applications of the Selberg integral, for instance in the theory of integrable systems (Chapters 8 and 9), in conformal field theory (Chapter 11) and in random matrix theory; see the overview article by Forrester and Warnaar (2008).

For basic hypergeometric integrals associated with root systems a similar story applies. The roles of Lie groups and root systems are taken over by quantum groups and affine root systems, although this time the representation-theoretic context came later. The *affine root system* associated to an irreducible reduced root system R is denoted by $R^{(1)}$ and consists of the collection of affine linear functionals $a\colon \mathbb{R}^n \to \mathbb{R}$ of the form $a(\boldsymbol{x}) = (\alpha, \boldsymbol{x}) + m$ (with $\alpha \in R$ and $m \in \mathbb{Z}$). The role of $w_{\boldsymbol{k}}(\boldsymbol{x})$ is now taken over by

$$w_{\boldsymbol{k},q}(\boldsymbol{x}) = \prod_{a\in R^{(1)};\, a(0)\ge 0} \left(\frac{1-q^{a(\boldsymbol{x})}}{1-q^{k_a + a(\boldsymbol{x})}} \right) = \prod_{\alpha\in R} \frac{(q^{(\alpha,\boldsymbol{x})}; q_\alpha)_\infty}{(q^{k_\alpha+(\alpha,\boldsymbol{x})}; q_\alpha)_\infty},$$

where $k_a = k_\alpha$ if α is the gradient of $a \in R^{(1)}$. Macdonald (1982) conjectured an explicit evaluation for the *basic hypergeometric integral*

$$\int_{A_R} w_{\boldsymbol{k},q}(\boldsymbol{x}/\tau)\, \mathrm{d}\boldsymbol{x}, \qquad q = \exp(2\pi i \tau) \tag{1.2.12}$$

associated with R, which was proved in full generality by Cherednik (1995) using the theory of double affine Hecke algebras. The evaluation formula gives the volume of the orthogonality measure of the Macdonald polynomials; see Chapter 9. The integral (1.2.12) and its evaluation generalize to arbitrary (possibly nonreduced) irreducible affine root systems and with milder equivariance conditions on $\boldsymbol{k} = \{k_a\}_{a\in R^{(1)}}$. In the case of the nonreduced affine root system of type $\mathrm{C}^\vee\mathrm{C}_n$, this leads to the multivariable analogue of the Askey–Wilson integral (Gustafson, 1990) which depends, apart from q, on five additional parameters. It gives the volume of the orthogonality measure of the Koornwinder polynomials; see Chapter 9.

A very general elliptic analogue of the Selberg integral and of Gustafson's multivariable analogue of the Askey–Wilson integral was conjectured by van Diejen and Spiridonov (2001, Theorem 4.2) and proved by Rains (2010, Theorem 6.1):

$$\begin{aligned}\frac{1}{(2\pi i)^n}\int_{T^n}\prod_{1\le i<j\le n}\frac{\Gamma(tz_iz_j,tz_iz_j^{-1},tz_i^{-1}z_j,tz_i^{-1}z_j^{-1})}{\Gamma(z_iz_j,z_iz_j^{-1},z_i^{-1}z_j,z_i^{-1}z_j^{-1})}\prod_{k=1}^{n}\frac{\prod_{i=1}^{6}\Gamma(t_iz_k,t_iz_k^{-1})}{\Gamma(z_k^2,z_k^{-2})}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}\\ =\frac{2^n n!}{(p;p)_\infty^n(q;q)_\infty^n}\prod_{m=1}^{n}\left(\frac{\Gamma(t^m)}{\Gamma(t)}\prod_{1\le i<j\le 6}\Gamma(t^{m-1}t_it_j)\right),\end{aligned} \tag{1.2.13}$$

with T the positively oriented unit circle in the complex plane, $\Gamma(x_1,\ldots,x_r) := \Gamma(x_1)\cdots\Gamma(x_r)$ a product of elliptic Gamma functions $\Gamma(x_i)$, and parameters $t, t_i \in \mathbb{C}$ satisfying $|t|, |t_i| < 1$ and $t^{2n-2}t_1\cdots t_6 = pq$. The integral (1.2.13) is an example of an *elliptic hypergeometric integral* associated with the root system of type C_n. For $n = 1$ it reduces to Spiridonov's elliptic beta integral (Spiridonov and Zhedanov, 2000). It is a special case of a family of transformation formulas that relate elliptic hypergeometric integrals associated with type-C root systems of different ranks (van de Bult, 2009). The basic analogue of (1.2.13) is a multivariable analogue of the *Nassrallah–Rahman integral* (Nassrallah and Rahman, 1985),

$$\frac{1}{2\pi i}\int_T\frac{(z^2,z^{-2},Az,Az^{-1};q)_\infty}{\prod_{j=1}^{5}(t_jz,t_jz^{-1};q)_\infty}\frac{dz}{z}=\frac{2\prod_{j=1}^{5}(At_j^{-1};q)_\infty}{(q;q)_\infty\prod_{1\le j<k\le 5}(t_jt_k;q)_\infty},\qquad |t_j|<1 \tag{1.2.14}$$

where $A := t_1t_2t_3t_4t_5$ and $(a_1,\ldots,a_r;q)_\infty := \prod_{i=1}^{r}(a_i;q)_\infty$. Just as (1.2.14) gives the Askey–Wilson integral for $t_5 = 0$, its multivariable analogue yields Gustafson's (1990) integral by the same substitution. The identity (1.2.13) and some of its degenerations give the volumes of (bi)orthogonality measures for important families of multivariable (bi)orthogonal functions; see §1.4.

The multivariable elliptic integrals appearing in Felder et al. (1997, 1999) as coordinates of vector-valued solutions of *qKZB* equations are associated with the root system of type A_n. Their semiclassical limits, which provide solutions of the *KZB* equation, as well as their degenerations to the basic and classical hypergeometric level, are discussed in Chapter 11.

A further rough division of hypergeometric integrals associated with root systems involves the notion of types. Multidimensional integrals are said to be *type-II basic* (resp. *elliptic*) *hypergeometric integrals* associated with the root system R if the integrand contains a factor of the form $\prod_{\alpha\in R}(\Gamma(q^{(\alpha,\boldsymbol{x})})/\Gamma(q^{k_\alpha+(\alpha,\boldsymbol{x})}))$ with $\Gamma(x)$ the basic (resp. elliptic) Gamma function. It is called *type I* if it contains a factor of the form $\prod_{\alpha\in R}\Gamma(q^{(\alpha,\boldsymbol{x})})^{-1}$. Similarly, a multidimensional integral is said to be a *type-II classical hypergeometric integral* associated with the root system R if the integrand contains a factor of the form $\Delta_{\boldsymbol{k}}(\boldsymbol{x})$ or $\prod_{\alpha\in R}(\Gamma(\alpha,\boldsymbol{x})/\Gamma(k_\alpha+(\alpha,\boldsymbol{x})))$, with $\Gamma(x)$ the classical Gamma function (and a similar adjustment for type I). The examples of multidimensional integral evaluations highlighted so far are type II. In Chapters 5 and 6 many examples of type-I and type-II multidimensional integral evaluations and transformations are discussed. Note that there are also hypergeometric integrals of mixed type; see (6.2.3) for an example.

Next we turn our attention to multivariable hypergeometric series. For a given root system R, we can define a *Weyl-type denominator* by

$$\Delta(\boldsymbol{x}) := \prod_{\alpha\in R^+} h((\alpha,\boldsymbol{x})) \quad \text{with } h(z) = \begin{cases} z & \text{(classical hypergeometric type)}, \\ 1-q^z & \text{(basic hypergeometric type)}, \\ \theta(z;p) & \text{(elliptic hypergeometric type)}. \end{cases}$$

In the basic hypergeometric case $\Delta(\boldsymbol{x})$ is the Weyl denominator of the semisimple Lie algebra associated with R, while in the elliptic case $\Delta(\boldsymbol{x})$ is closely related to the Weyl denominator of the affine Lie algebra associated with $R^{(1)}$; see §6.1.2.

Multivariable classical, basic and elliptic hypergeometric series $\sum_{\boldsymbol{k}\in D} f(\boldsymbol{k})$ $(D \subseteq \mathbb{Z}^n)$ are said to be *associated with the classical root system* R if $f(\boldsymbol{k})$ contains the factor $\Delta(\boldsymbol{y}+\boldsymbol{k})$ for some fixed $\boldsymbol{y} \in \mathbb{C}^n$ in a nontrivial way. First examples of multivariable classical hypergeometric series identities appeared in the work of Holman et al. (1976) on $6j$-symbols for $\mathrm{SU}(n)$ (the associated root system is of type A). An important nontrivial example of a multivariable basic hypergeometric series identity is the fundamental theorem of Milne (1985):

$$\sum_{\boldsymbol{k}\in D_N} \Delta(\boldsymbol{y}+\boldsymbol{k}) \prod_{\ell=1}^{n} q^{(\ell-1)k_\ell} \prod_{i,j=1}^{n} \frac{(q^{\beta_i+y_j-y_i};q)_{k_j}}{(q^{1+y_j-y_i};q)_{k_j}} = \frac{(q^{\beta_1+\cdots+\beta_n};q)_N}{(q;q)_N}\Delta(\boldsymbol{y})$$

with $D_N := \{\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0} \mid k_1+\cdots+k_n = N\}$ and $\Delta(\boldsymbol{x}) = \prod_{1\leq i<j\leq n}(1-q^{x_i-x_j})$ the Weyl-type denominator for the root system R of type A_{n-1}. An elliptic generalization is the elliptic Jackson summation formula (6.3.1a) due to Rosengren (2004, Theorem 5.1).

For classical root systems, identities and transformations for multivariable hypergeometric series naturally arise from related multidimensional hypergeometric integral identities and transformations through residue calculus. In this process, the Weyl denominator $\Delta(\boldsymbol{k})$ arises from the integrands of the multidimensional hypergeometric integrals through the formula (6.1.3). The residue calculus typically involves iterated small contour deformations per coordinate, avoiding at each step the poles of the factors of the integrand that do not depend on a single coordinate x_j. This technique was developed in Stokman (2000), where it was applied to type-II basic hypergeometric integrals associated with Koornwinder polynomials. When applied to the elliptic Selberg integral (1.2.13) one obtains a type-C elliptic hypergeometric series identity (see (6.3.6)) that reduces for $n = 1$ to the *Frenkel–Turaev elliptic summation formula* (Frenkel and Turaev, 1997):

$$\begin{aligned}\sum_{m=0}^{N} \frac{\theta(aq^{2m};p)}{\theta(a;p)} \frac{(a,b,c,d,e,q^{-N};q,p)_m}{(q,aq/b,aq/c,aq/d,aq/e,aq^{1+N};q,p)_m} q^m \\ = \frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_N}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_N}\end{aligned} \tag{1.2.15}$$

for $bcde = a^2q^{N+1}$, where $(x_1,\ldots,x_r;q,p)_m = \prod_{i=1}^r (x_i;q,p)_m$. In this way, many of the hypergeometric series identities and transformations associated with classical root systems as discussed in Chapter 5 (classical and basic hypergeometric) and in Chapter 6 (elliptic hypergeometric) can be viewed as discrete versions of multidimensional hypergeometric integrals.

1.3 Multivariable (Bi)Orthogonal Polynomials and Functions

1.3.1 One-Variable Cases

The class of one-variable (bi)orthogonal polynomials and functions splits up into various natural subclasses, each subclass having its distinct features that are vital for the construction and study of its multivariable generalization.

(a) General theory of orthogonal polynomials (Szegő, 1975)
(b) Classical orthogonal polynomials
(c) Classical biorthogonal rational functions
(d) Bessel functions (Olver et al., 2010, §10.22(v)) and Jacobi functions (Koornwinder, 1984)

By *classical orthogonal polynomials* we mean (more generally than in Chapter 2) the one-variable orthogonal polynomials belonging to the Askey or q-Askey scheme (Koekoek et al., 2010, Chs. 9, 14). They are characterized as the orthogonal polynomials that are joint eigenfunctions of a suitable type of second-order differential or (q-)difference operator. The corresponding classification results are called (generalized) Bochner theorems (Grünbaum and Haine, 1996; Ismail, 2003; Vinet and Zhedanov, 2008). Prominent members are the Jacobi polynomials (Szegő, 1975, Ch. IV) and their top-level q-analogues, the Askey–Wilson polynomials (Askey and Wilson, 1985). By *classical biorthogonal rational functions* we refer to the generalizations of classical orthogonal polynomials due to Rahman (1986, 1991) and Wilson (1991), and their elliptic analogues due to Spiridonov and Zhedanov (2000).

Classical orthogonal polynomials and biorthogonal rational functions are expressible as ordinary, basic and elliptic hypergeometric series. The various classes admit (bi)orthogonality relations with respect to explicit measures whose total masses are the outcome of important integral evaluation formulas. For example, for the classical Jacobi polynomials the integral evaluation is the beta integral

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x=\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},\qquad \alpha,\beta>-1,$$

with $\Gamma(x)$ the classical Gamma function, for Rahman's (1986) biorthogonal basic hypergeometric rational functions it is the Nassrallah–Rahman integral (1.2.14) and for Spiridonov–Zhedanov's (2000) elliptic biorthogonal rational functions it is Spiridonov's (2001) elliptic beta integral (the $n = 1$ case of (1.2.13)).

1.3.2 Multivariable Generalizations

The subclasses (a)–(d) of one-variable (bi)orthogonal polynomials and functions generalize to the multivariable case as follows. Note that the one-variable subclass (b) generalizes to two subclasses (b1) and (b2).

(a) General theory of multivariable orthogonal polynomials with respect to orthogonality measures on $\mathbb{R}^d$ (Dunkl and Xu, 2014). This is discussed in Chapter 2.
(b1) Multivariable orthogonal polynomials expressible as (nonstraightforward) products of one-variable classical orthogonal polynomials and elementary polynomials; see