Portfolio Theory and Risk Management

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To Anna, Emily, Staś, Weronika and Helenka

Contents

Preface

In this fifth volume of the series 'Mastering Mathematical Finance' we present a self-contained rigorous account of mean-variance portfolio theory, as well as a simple introduction to utility functions and modern risk measures.

Portfolio theory, exploring the optimal allocation of wealth among different assets in an investment portfolio, based on the twin objectives of maximising return while minimising risk, owes its mathematical formulation to the work of Harry Markowitz[1] in 1952; for which he was awarded the Nobel Prize in Economics in 1990. Mean-variance analysis has held sway for more than half a century, and forms part of the core curriculum in financial economics and business studies. In these settings mathematical rigour may suffer at times, and our aim is to provide a carefully motivated treatment of the mathematical background and content of the theory, assuming only basic calculus and linear algebra as prerequisites.

Chapter 1 provides a brief review of the key concepts of return and risk, while noting some defects of variance as a risk measure. Considering a portfolio with only two risky assets, we show in Chapter 2 how the minimum variance portfolio, minimum variance line, market portfolio and capital market line may be found by elementary calculus methods. Chapter 3 contains a careful account of the method of Lagrange multipliers, including a discussion of sufficient conditions for extrema in the special case of quadratic forms. These techniques are applied in Chapter 4 to generalise the formulae obtained for two-asset portfolios to the general case.

The derivation of the Capital Asset Pricing Model (CAPM) follows in Chapter 5, including two proofs of the CAPM formula, based, respectively, on the underlying geometry (to elucidate the role of beta) and linear algebra (leading to the security market line), and introducing performance measures such as the Jensen index and Sharpe ratio. The security characteristic line is shown to aid the least-squares estimation of beta using historical portfolio returns and the market portfolio.

Chapter 6 contains a brief introduction to utility theory. To keep matters simple we restrict to finite sample spaces to discuss preference relations.

[1] H. Markowitz, Portfolio selection, *Journal of Finance* 7 (1), (1952), 77–91.

We consider examples of von Neumann–Morgenstern utility functions, link utility maximisation with the No Arbitrage Principle and explain the key role of state price vectors. Finally, we explore the link between utility maximisation and the CAPM and illustrate the role of the certainty equivalent for the risk averse investor.

In the final two chapters the emphasis shifts from variance to measures of downside risk. Chapter 7 contains an account of Value at Risk (VaR), which remains popular in practice despite its well-documented shortcomings. Following a careful look at quantiles and the algebraic properties of VaR, our emphasis is on computing VaR, especially for assets within the Black–Scholes framework. A novel feature is an account of VaR-optimal hedging with put options, which is shown to reduce to a linear programming problem if the parameters are chosen with care.

In Chapter 8 we examine how the defects of VaR can be addressed using coherent risk measures. The principal example discussed is Average Value at Risk (AVaR), which is described in detail, including a careful proof of sub-additivity. AVaR is placed in the context of coherent risk measures, and generalised to yield spectral risk measures. The analysis of hedging with put options in the Black–Scholes setting is revisited, with AVaR in place of VaR, and the outcomes are compared in examples.

Throughout this volume the emphasis is on examples, applications and computations. The underlying theory is presented rigorously, but as simply as possible. Proofs are given in detail, with the more demanding ones left to the end of each chapter to avoid disrupting the flow of ideas. Applications presented in the final chapters make use of background material from the earlier volumes [PF] and [BSM] in the current series. The exercises form an integral part of the volume, and range from simple verification to more challenging problems. Solutions and additional material can be found at www.cambridge.org/9781107003675, which will be updated regularly.