1

Risk and return

Financial investors base their activity on the expectation that their investment will increase over time, leading to an increase in wealth. Over a fixed time period, the investor seeks to maximise the return on the investment, that is, the increase in asset value as a proportion of the initial investment. The final values of most assets (other than loans at a fixed rate of interest) are uncertain, so that the returns on these investments need to be expressed in terms of random variables. To estimate the return on such an asset by a single number it is natural to use the expected value of the return, which averages the returns over all possible outcomes.

Our uncertainty about future market behaviour finds expression in the second key concept in finance: risk. Assets such as stocks, forward contracts and options are risky because we cannot predict their future values with certainty. Assets whose possible final values are more ‘widely spread’ are naturally seen as entailing greater risk. Thus our initial attempt to measure the riskiness of a random variable will measure the spread of the return, which rational investors will seek to minimise while maximising their return.

In brief, return reflects the efficiency of an investment, risk is concerned with uncertainty. The balance between these two is at the heart of portfolio theory, which seeks to find optimal allocations of the investor’s initial wealth among the available assets: maximising return at a given level of risk and minimising risk at a given level of expected return.

1.1 Expected return

We are concerned with just two time instants: the present time, denoted by 0, and the future time 1, where 1 may stand for any unit of time. Suppose we make a single-period investment in some stock with the current price $S(0)$ known, and the future price $S(1)$ unknown, hence assumed to be represented by a random variable

$$S(1) : \Omega \to [0, +\infty),$$

where Ω is the sample space of some probability space $(\Omega, \mathcal{F}, P)$. The members of Ω are often called **states** or **scenarios**. (See [PF] for basic definitions.)

When Ω is finite, $\Omega = \{\omega_1, \ldots, \omega_N\}$, we shall adopt the notation

$$S(1, \omega_i) = S(1)(\omega_i) \quad \text{for } i = 1, \ldots, N,$$

for the possible values of $S(1)$. In this setting it is natural to equip Ω with the σ-field $\mathcal{F} = 2^\Omega$ of all its subsets. To define a probability measure $P : \mathcal{F} \to [0, 1]$ it is sufficient to give its values on single element sets, $P(\{\omega_i\}) = p_i$, by choosing $p_i \in (0, 1]$ such that $\sum_{i=1}^N p_i = 1$. We can then compute the expected price at the end of the period

$$\mathbb{E}(S(1)) = \sum_{i=1}^N S(1, \omega_i) p_i,$$

and the variance of the price

$$\mathrm{Var}(S(1)) = \sum_{i=1}^N (S(1, \omega_i) - \mathbb{E}(S(1)))^2 \, p_i.$$

Example 1.1
Assume that $S(0) = 100$ and

$$S(1) = \begin{cases} 120 & \text{with probability } \frac{1}{2}, \\ 90 & \text{with probability } \frac{1}{2}. \end{cases}$$

Then $\mathbb{E}(S(1)) = \frac{1}{2}120 + \frac{1}{2}90 = 105$ and $\mathrm{Var}(S(1)) = (120 - 105)^2 \frac{1}{2} + (90 - 105)^2 \frac{1}{2} = 15^2$. Observe also that the standard deviation, which is the square root of the variance, is equal to $\sqrt{\mathrm{Var}(S(1))} = 15$.

Exercise 1.1 Assume that $U, D \in \mathbb{R}$ are such that $-1 < D < U$. Assume also that S has a binomial distribution, that is

$$P\left(S(1) = S(0)(1+U)^k(1+D)^{N-k}\right) = \binom{N}{k} p^k (1-p)^{N-k},$$

for $k \in \{0, 1, \ldots, N\}$. Compute $\mathbb{E}(S(1))$ and $\mathrm{Var}(S(1))$.

When $S(1)$ is continuously distributed, with density function $f : \mathbb{R} \to \mathbb{R}$, then

$$\mathbb{E}(S(1)) = \int_{-\infty}^{\infty} x f(x) dx,$$

and

$$\mathrm{Var}(S(1)) = \int_{-\infty}^{\infty} (x - \mathbb{E}(S(1)))^2 f(x) dx.$$

Example 1.2

Assume that $S(1) = S(0)\exp(m + sZ)$, where Z is a random variable with standard normal distribution $N(0,1)$. This means that $S(1)$ has lognormal distribution. The density function of $S(1)$ is equal to

$$f(x) = \frac{1}{xs\sqrt{2\pi}} e^{-\frac{\left(\ln \frac{x}{S(0)} - m\right)^2}{2s^2}} \quad \text{for } x > 0,$$

and 0 for $x \leq 0$. We can compute the expected price as

$$\begin{aligned}
\mathbb{E}(S(1)) &= \int_0^{\infty} x f(x) dx \\
&= \int_0^{\infty} \frac{1}{s\sqrt{2\pi}} e^{-\frac{\left(\ln \frac{x}{S(0)} - m\right)^2}{2s^2}} dx \\
&= \int_{-\infty}^{\infty} S(0) e^{sy+m} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \qquad \left(\text{taking } y = \frac{1}{s}\left(\ln \frac{x}{S(0)} - m\right)\right) \\
&= S(0) e^{m+\frac{s^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-s)^2}{2}} dy \\
&= S(0) e^{m+\frac{s^2}{2}}.
\end{aligned}$$

Exercise 1.2 Consider $S(1)$ from Example 1.2. Show that

$$\mathrm{Var}(S(1)) = S(0)^2\left(e^{s^2}-1\right)e^{2m+s^2}.$$

While we may allow any probability space, we must make sure that negative values of the random variable $S(1)$ are excluded since negative prices make no sense from the point of view of economics. This means that the distribution of $S(1)$ has to be supported on $[0,+\infty)$ (meaning that $P(S(1)\geq 0)=1$).

The **return** (also called the rate of return) on the investment S is a random variable $K:\Omega\to\mathbb{R}$, defined as

$$K=\frac{S(1)-S(0)}{S(0)}.$$

By the linearity of mathematical expectation, the **expected** (or mean) **return** is given by

$$\mathbb{E}(K)=\frac{\mathbb{E}(S(1))-S(0)}{S(0)}.$$

We introduce the convention of using the Greek letter μ for expectations of various random returns

$$\mu=\mathbb{E}(K),$$

with various subscripts indicating the context, if necessary.

The relationships between the prices and returns can be written as

$$S(1)=S(0)(1+K),$$
$$\mathbb{E}(S(1))=S(0)(1+\mu),$$

which illustrates the possibility of reversing the approach: given the returns we can find the prices.

The requirement that $S(1)$ is nonnegative implies that we must have $K\geq -1$. This in particular excludes the possibility of considering K with Gaussian (normal) distribution.

At time 1 a dividend may be paid. In practice, after the dividend is paid, the stock price drops by this amount, which is logical. Thus we have to determine the price that includes the dividend; more precisely, we must distinguish between the right to receive that price (the cum dividend price) and the price after the dividend is paid (the ex dividend price). We assume that $S(1)$ denotes the latter, hence the definition of the return has to be modified to account for dividends:

$$K = \frac{S(1) + \mathrm{Div}(1) - S(0)}{S(0)}.$$

A **bond** is a special security that pays a certain sum of money, known in advance, at maturity; this sum is the same in each state. The return on a bond is not random (recall that we are dealing with a single time period). Consider a bond paying a unit of home currency at time 1, that is $B(1) = 1$, which is purchased for $B(0) < 1$. Then

$$R = \frac{1 - B(0)}{B(0)}$$

defines the **risk-free return**. The bond price can be expressed as

$$B(0) = \frac{1}{1+R},$$

giving the present value of a unit at time 1.

Exercise 1.3 Compute the expected returns for the stocks described in Exercise 1.1 and Example 1.2.

Exercise 1.4 Assume that $S(0) = 80$ and that the ex dividend price is

$$S(1) = \begin{cases} 60 & \text{with probability } \frac{1}{6}, \\ 80 & \text{with probability } \frac{3}{6}, \\ 90 & \text{with probability } \frac{2}{6}. \end{cases}$$

The company will pay out a constant dividend (independent of the future stock price). Compute the dividend for which the expected return on stock would be 20%.

1.2 Variance as a risk measure

The concept of risk in finance is captured in many ways. The basic and most widely used one is concerned with risk as uncertainty of the unknown future value of some quantity in question (here we are concerned with return). This uncertainty is understood as the scatter around some reference point. A natural candidate for the reference value is the mathematical expectation (though other benchmarks are sometimes considered). The extent of scatter is conveniently measured by the variance. This notion takes care of two aspects of risk:

(i) The distances between possible values and the expectation.
(ii) The probabilities of attaining the various possible values.

Definition 1.3
By (the measure of) risk we mean the **variance** of the return

$$\mathrm{Var}(K) = \mathbb{E}(K-\mu)^2 = \mathbb{E}(K^2) - \mu^2,$$

or the **standard deviation** $\sqrt{\mathrm{Var}(K)}$.

The variance of the return can be computed from the variance of $S(1)$,

$$\begin{aligned}\mathrm{Var}(K) &= \mathrm{Var}\left(\frac{S(1)-S(0)}{S(0)}\right)\\ &= \frac{1}{S(0)^2}\mathrm{Var}\left(S(1)-S(0)\right)\\ &= \frac{1}{S(0)^2}\mathrm{Var}\left(S(1)\right).\end{aligned}$$

We use the Greek letter σ for standard deviations of various random returns

$$\sigma = \sqrt{\mathrm{Var}(K)},$$

qualified by subscripts, as required.

Exercise 1.5 In a market with risk-free return $R > 0$, we buy a 'leveraged' stock S at time 0 with a mixture of cash and a loan at rate R. To buy the stock for $S(0)$ we use $wS(0)$ of our own cash and borrow $(1-w)S(0)$, for some $w \in (0,1)$. Denote the returns at time 1 on the stock and leveraged position by K_S and K_{lev} respectively.

Derive the relation

$$K_{\text{lev}} = R + \frac{1}{w}(K_S - R),$$

and find the relationship between the standard deviations of the stock and the leveraged position.

Standard deviation alone does not fully capture the risk of an investment. We illustrate this with a simple example.

Example 1.4
Consider three assets with today's prices $S_i(0) = 100$ for $i = 1, 2, 3$ and time 1 prices with the following distributions:

$$S_1(1) = \begin{cases} 120 & \text{with probability } \frac{1}{2}, \\ 90 & \text{with probability } \frac{1}{2}, \end{cases}$$

$$S_2(1) = \begin{cases} 140 & \text{with probability } \frac{1}{2}, \\ 90 & \text{with probability } \frac{1}{2}, \end{cases}$$

$$S_3(1) = \begin{cases} 130 & \text{with probability } \frac{1}{2}, \\ 100 & \text{with probability } \frac{1}{2}. \end{cases}$$

We can see that

$$\sigma_1 = \sqrt{\text{Var}(K_1)} = 0.15,$$
$$\sigma_2 = \sqrt{\text{Var}(K_2)} = 0.25,$$
$$\sigma_3 = \sqrt{\text{Var}(K_2)} = 0.15.$$

Here $\sigma_2 > \sigma_1$ and $\sigma_3 = \sigma_1$, but both the second and third assets are preferable to the first, since at time 1 they bring in more cash. We shall return to this example in the next section.

When considering the risk of an investment we should take into account both the expectation and and the standard deviation of the return. Given the choice between two securities a rational investor will, if possible, choose that with the higher expected return and lower standard deviation, that is, lower risk. This motivates the following definition.

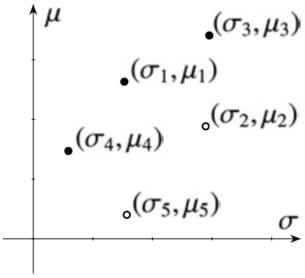

Figure 1.1 Efficient subset.

Definition 1.5
We say that a security with expected return μ_1 and standard deviation σ_1 **dominates** another security with expected return μ_2 and standard deviation σ_2 whenever

$$\mu_1 \geq \mu_2 \quad \text{and} \quad \sigma_1 \leq \sigma_2.$$

The meaning of the word 'dominates' is that we assume the investors to be risk averse. One can imagine an investor whose personal goal is just the excitement of playing the market. This person will not pay any attention to return or may prefer higher risk. However, it is not our intention to cover such individuals by our theory.

The playground for portfolio theory will be the (σ, μ)-plane, in fact the right half-plane since the standard deviation is non-negative. Each security is represented by a dot on this plane. This means that we are making a simplification by assuming that the expectation and variance are all that matters when investment decisions are made.

We assume that the dominating securities are preferred, which geometrically (geographically) means that for any two securities, the one lying further north-west in the (σ, μ)-plane is preferable. This ordering does not allow us to compare all pairs: in Figure 1.1 we see for instance that the pairs (σ_1, μ_1) and (σ_3, μ_3) are not comparable.

Given a set A of securities in the (σ, μ)-plane, we consider the subset of all maximal elements with respect to the dominance relation and call it the **efficient subset**. If the set A is finite, finding the efficient subsets reduces to eliminating the dominated securities. Figure 1.1 shows a set of five securities with efficient subset consisting of just three, numbered 1, 3 and 4.

Exercise 1.6 Assume that we have three assets. The first has expected return $\mu_1 = 10\%$ and standard deviation of return equal to $\sigma_1 = 0.25$. The second has expected return $\mu_2 = 15\%$ and standard deviation of return equal to $\sigma_2 = 0.3$. Assume that the future prices of the third asset will have $\mathbb{E}(S_3(1)) = 100$, $\sqrt{Var(S_3(1))} = 20$. Find the ranges of prices $S_3(0)$ so that the following conditions are satisfied:

(i) The third asset dominates the first asset.
(ii) The third asset dominates the second asset.
(iii) No asset is dominated by another asset.

1.3 Semi-variance

Consider the three assets described in Example 1.4. Although $\sigma_1 = \sigma_3$, the third asset carries no 'downside risk', since neither outcome for $S_3(1)$ involves a loss for the investor. Similarly, although $\sigma_2 > \sigma_1$, the downside risk for the second asset is the same as that for the first (a 50% chance of incurring a loss of 10), but the expected return for the second asset is 15%, making it the more attractive investment even though, as measured by variance, it is more risky. Since investors regard risk as concerned with failure (i.e. downside risk), the following modification of variance is sometimes used. It is called **semi-variance** and is computed by a formula that takes into account only the unfavourable outcomes, where the return is below the expected value

$$\mathbb{E}(\min\{0, K - \mu\})^2. \tag{1.1}$$

The square root of semi-variance is denoted by semi-σ. However, this notion still does not agree fully with the intuition.

Example 1.6
Assume that $\Omega = \{\omega_1, \omega_2\}$, $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$ and

$$K(\omega_1) = 10\%,$$
$$K(\omega_2) = 20\%.$$

Consider a modification K' with

$$K'(\omega_1) = 10\%,$$
$$K'(\omega_2) = 30\%.$$

Then K' is definitely better than K but the semi-variance and the variance for K' are both higher than for K.

If variance or semi-variance are to represent risk, it is illogical that a better version should be regarded as more risky. This defect can be rectified by replacing the expectation by some other reference point, for instance the risk-free return with the following modification of (1.1),

$$\mathbb{E}(\min\{0, K - R\})^2,$$

which eliminates the above unwanted feature. Instead of the risk-free rate, one can also consider the return required by the investor.

These versions are not very popular in the financial world, the variance being the basic measure of risk. In our presentation of portfolio theory we follow the historical tradition and take variance as the measure of risk. It is possible to develop a version of the theory for alternative ways of measuring risk. In most cases, however, such theories do not produce neat analytic formulae as is the case for the mean and variance.

We will return to a more general discussion of risk measures in the final chapters of this volume. An analysis of the popular concept of Value at Risk (VaR), which has been used extensively in the banking and investment sectors since the 1990s, will lead us to conclude that, despite its ubiquity, this risk measure has serious shortcomings, especially when dealing with mixed distributions. We will then examine an alternative which remedies these defects but still remains mathematically tractable.