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Introduction

This volume introduces simple mathematical models of financial markets, focussing on the problems of pricing and hedging risky financial instruments whose price evolution depends on the prices of other risky assets, such as stocks or commodities. Over the past four decades trading in these **derivative securities** (so named since their value derives from those of other, **underlying**, assets) has expanded enormously, not least as a result of the availability of mathematical models that provide initial pricing benchmarks. The markets in these financial instruments have provided investors with a much wider choice of investment vehicles, often tailor-made to specific investment objectives, and have led to greatly enhanced liquidity in asset markets. At the same time, the proliferation of ever more complex derivatives has led to increased market volatility resulting from the search for ever-higher short-term returns, while the sheer speed of expansion has made investment banking a highly specialised business, imperfectly understood by many investors, boards of directors and even market specialists. The consequences of 'irrational exuberance' in some markets have been brought home painfully by stock market crashes and banking crises, and have led to increased regulation.

It seems to us a sound principle that market participants should have a clear understanding of the products they trade. Thus a better grasp of the basic modelling tools upon which much of modern derivative pricing is based is essential. These tools are mathematical techniques, informed by some basic economic precepts, which lead to a clearer formulation and quantification of the risk inherent in a given transaction, and its impact on possible returns.

The formulation and use of dynamical models, employing stochastic calculus techniques or partial differential equations to describe and analyse market models and price various instruments, requires substantial mathematical preparation. The same is not true of the simpler discrete-time models that we discuss in this volume. Only a modicum of probability theory is required for their understanding, and we gradually introduce the main ingredients, trying to make the volume self-contained, although familiarity with basic probability may give the reader some advantage.

Discrete-time models, using finitely many possible trading dates and a finite collection of possible outcomes (scenarios), have intuitive appeal, since the world we inhabit is finite and, as Keynes famously observed: 'in the long run we are all dead'. Moreover, even the simplest models already embody the principal features of the modelling techniques needed in more complex settings, and it may be easier to 'see the wood for the trees' in the friendlier environment of a simple model. At the same time, one must recognise that models are only useful to the extent that they agree with the observed data; hence the need to reflect actual market practice more accurately must inevitably introduce ever greater complexity.

These more realistic models, and the necessary mathematical background to understand them fully, will be introduced and studied in the companion volumes in this series.

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Single-step asset pricing models

In this chapter we explore the simplest option pricing models. We assume that there is a single trading period, from the present date (time 0) to a fixed future time T. We examine mathematical models for the behaviour of the prices of one or more underlying stocks, beginning with the simplest case of a single stock whose price at time T takes one of just two possible values. This analysis is extended to a model with two stocks, models with three possible prices for each stock, and finally to a general pricing model with $d > 1$ stocks, each with m possible outcomes for its price at T.

The principal economic requirement, arising from the assumption that the financial markets being modelled are efficient is that, provided all market participants share the same information about the random evolution of the stock price, an investor in this market should not be able to make a profit without incurring some level of risk. This assumption is cast in mathematical terms and underlies the methods we develop to establish how to determine the value of financial instruments whose price depends only on the price of the underlying stock at time T: the so-called European derivative securities, whose study is the principal topic of this and the next two chapters.

2.1 Single-step binomial tree

The most important feature of financial markets is uncertainty arising from the fact that the future prices of various financial assets cannot be predicted. We begin by examining the simplest possible setting and construct a 'toy model' whose two dimensions – time and uncertainty – are illustrated by means of the simplest possible tools.

We take time as discrete and reduced to just two time instants: the present 0 and the future T. To simplify the notation and emphasise the fact that we are now dealing with one step we will write 1 instead of T, so we have in mind a single step of length T.

The uncertainty is reflected by the number of possible scenarios, which in this section is minimal, namely two: 'up' or 'down'. This deceptively simple model reflects many of the key features of much more general pricing models, and we examine it in detail.

Assets

At time 0 assume we are given some asset, which is usually thought of as a stock and is customarily referred to, in defiance of proper grammatical usage, as **the underlying** – since its price will determine the prices of the securities we wish to study.

The current price $S(0) > 0$ of the underlying is known, while its future price $S(1)$ is not known, but we consider possible future prices and the probabilities of attaining them. This is performed formally by first choosing a set Ω, called a **sample space**. The elements of this set are related to outcomes of some random experiment (either specific, tossing a coin, or quite vague, the future state of the economy) and we call them **scenarios** or **states**. In this volume Ω will always be a finite set.

Just for the current section we take $\Omega = \{\mathrm{u}, \mathrm{d}\}$ and we let the future price of our asset be a function

$$S(1)\colon \Omega \to (0, +\infty).$$

In general, a function defined on Ω is called a **random variable**. Since $S(1)$ takes just two values, so does the **return**

$$K_S = \frac{S(1) - S(0)}{S(0)},$$

which determines the price

$$S(1) = S(0)(1 + K_S)$$

and we will define $S(1)$ by first deciding the shape of K_S. Assume that

$$K_S(\mathrm{u}) = U,$$
$$K_S(\mathrm{d}) = D,$$

where

$$-1 < D < U$$

to avoid negative prices. Then

$$S(1,\mathrm{u}) = S(0)(1+U),$$
$$S(1,\mathrm{d}) = S(0)(1+D).$$

It is convenient to introduce the following notational convention: $S(1,\mathrm{u}) = S^{\mathrm{u}}$, $S(1,\mathrm{d}) = S^{\mathrm{d}}$, which gives the random variable of stock prices as

$$S(1) = \begin{cases} S^{\mathrm{u}} = S(0)(1+U), \\ S^{\mathrm{d}} = S(0)(1+D). \end{cases}$$

The probabilities are given by prescibing one number p from the interval $(0,1)$ writing

$$P(\{\mathrm{u}\}) = p,$$
$$P(\{\mathrm{d}\}) = 1-p.$$

Here we are careful with notation since P assigns numbers to subsets of Ω, but the following simplified version is quite useful: $P(\{\omega\}) = P(\omega)$ for $\omega \in \Omega$. Furthermore, the probability P is assumed to be zero for the empty set and 1 for the whole Ω and so it is defined for all subsets of Ω, called **events**, satisfying additionally the so-called **additivity** property: for any $A, B \subset \Omega$, such that $A \cap B = \emptyset$,

$$P(A \cup B) = P(A) + P(B).$$

(The choice of such A, B in our case is rather limited!)

The accompanying non-random asset is the money market account with deterministic prices $A(0)$, $A(1)$, where

$$A(1) = A(0)(1+R)$$

for some $R > 0$.

Investment

Suppose we have at our disposal a certain sum of money, which we invest by purchasing a portfolio of x units of the above risky asset and y units of the money market account. We build a portfolio (x, y) with initial value given by

$$V_{(x,y)}(0) = xS(0) + yA(0).$$

The value of this portfolio at time 1 is

$$V_{(x,y)}(1) = xS(1) + yA(1),$$

that is

$$V_{(x,y)}(1) = \begin{cases} xS(0)(1+U) + yA(0)(1+R), \\ xS(0)(1+D) + yA(0)(1+R). \end{cases}$$

We assume that the market is **frictionless:** we do not impose any restrictions on the numbers x, y, so that unlimited **short-selling** is allowed: at time 0 we can borrow a share, sell it, and purchase some other asset, while at time 1 we buy the share back to return it to the owner. The assets are assumed to be arbitrarily **divisible,** meaning that x, y can take arbitrary real values. We do not impose any bounds on x, y, thus assuming unlimited **liquidity** in the market. Finally, there are no transaction costs involved in trading, i.e. the same stock price applies to **long** (buy: $x > 0$) and **short** (sell: $x < 0$) positions. Moreover, risk-free investment ($y > 0$) and borrowing ($y < 0$) both use the rate of return R.

Our most important modelling assumption is the No Arbitrage Principle (NAP), which asserts that trading cannot yield riskless profits. In general, the form of this assumption depends crucially on the class of trading strategies that we allow in the market. In the present context, with no intervening trading dates, this is trivial, since a 'strategy' consists of a single portfolio chosen at time 0.

Definition 2.1
A portfolio (x, y), chosen at time 0, is an **arbitrage** opportunity if $V_{(x,y)}(0) = 0$, $V_{(x,y)}(1) \geq 0$, and with positive probability $V_{(x,y)}(1) > 0$.

The last statement for our choice of Ω reduces to saying that we have strict inequality for at least one of the two possible scenarios.

Assumption 2.2 No Arbitrage Principle
Arbitrage opportunities do not exist in the market.

This has an immediate consequence for the relationship between the returns on the risky and riskless assets:

Theorem 2.3
The No Arbitrage Principle implies that

$$D < R < U.$$

Proof If $R \leq D$, that is

$$S(0)(1+R) \leq S^{\mathrm{d}} < S^{\mathrm{u}},$$

we put $x = 1$, $y = -S(0)/A(0)$. At time 1

$$V_{(x,y)}(1) = S(1) - S(0)(1+R) \geq 0,$$

which is strictly positive in the 'up' state, i.e. with probability $p > 0$.

If $R \geq U$, that is

$$S^{\mathrm{d}} < S^{\mathrm{u}} \leq S(0)(1+R),$$

we take the opposite position: $x = -1$, $y = S(0)/A(0)$ and

$$V_{(x,y)}(1) = -S(1) + S(0)(1+R) \geq 0$$

and this is strictly positive with probability $1 - p > 0$. In each case we have constructed a portfolio that leads to arbitrage. Hence neither inequality can hold and the theorem is proved. □

The converse implication is also true.

Theorem 2.4
The condition $D < R < U$ implies the No Arbitrage Principle.

Proof Suppose (x, y) is a portfolio with zero initial value. By definition of portfolio value

$$V_{(x,y)} = 0 = xS(0) + yA(0),$$
$$y = -x\frac{S(0)}{A(0)}.$$

We compute the terminal value with y as determined above:

$$\begin{aligned} V_{(x,y)}(1) &= xS(1) + yA(0)(1+R) \\ &= x(S(1) - S(0)(1+R)), \end{aligned}$$

thus

$$V_{(x,y)}(1) = \begin{cases} x[S^{\mathrm{u}} - S(0)(1+R)] = xS(0)(U-R) > 0 \\ x[S^{\mathrm{d}} - S(0)(1+R)] = xS(0)(D-R) < 0 \end{cases}$$

for any positive x, under the given assumption that $D < R < U$. If $x < 0$ the above signs are reversed and they remain opposites. So (x, y) cannot be an arbitrage for any choice of x. □

Definition 2.5
Suppose the asset X has value $X(1)$ at time 1. Its **discounted** value (to time 0) is given by

$$\tilde{X}(1) = X(1)(1+R)^{-1}.$$

For the stock S we have

$$\tilde{S}(1) = S(1)(1+R)^{-1},$$

and define the discounted stock price process $\tilde{S} = \{\tilde{S}(0), \tilde{S}(1)\}$ by also setting $\tilde{S}(0) = S(0)$. Clearly $\tilde{A}(1) = A(0)$. Similarly the discounted value process becomes

$$\tilde{V}_{(x,y)}(1) = V_{(x,y)}(1)(1+R)^{-1} = x\tilde{S}(1) + yA(0),$$
$$\tilde{V}_{(x,y)}(0) = V_{(x,y)}(0) = xS(0) + yA(0),$$

yielding

$$\begin{aligned}\tilde{V}_{(x,y)}(1) - \tilde{V}_{(x,y)}(0) &= x\tilde{S}(1) + yA(0) - (xS(0) + yA(0)) \\ &= x[\tilde{S}(1) - \tilde{S}(0)],\end{aligned}$$

so the change of the discounted portfolio value results exclusively from the change of the discounted value of the risky asset.

2.2 Option pricing

A **European call option** provides profit for the holder of a long position if the price of the underlying at time 1 is above a predetermined level K, the **strike** or **exercise** price, but eliminates the possibility of losses. We define the **payoff** of a call as

$$\begin{aligned}C(1) &= \begin{cases} S(1) - K & \text{if } S(1) > K, \\ 0 & \text{otherwise,}\end{cases} \\ &= (S(1) - K)^+,\end{aligned}$$

where we use the notation $f^+(\omega) = \max\{0, f(\omega)\}$ for any function f on Ω and $\omega \in \Omega$. (The reader should take care not to confuse the return K_S on the stock S with the strike price K of the option.) The effect of holding a call option is the right (without the obligation) to buy the underlying at the **expiry** time 1 for a price that cannot exceed K since if the asset is more expensive, the option pays the difference. If the underlying can be bought in the market for less than K at time 1 the option is simply not exercised. The right to sell the asset for at least K is called a (European) **put option** and its payoff is of the form $P(1) = (K - S(1))^+$. It is clear from the No Arbitrage Principle that options cannot be issued for free at time 0, since the payoffs are always non-negative and can have positive probability of being strictly positive. Finding 'rational' prices $C(0)$ (resp. $P(0)$) for a call (resp. put) option in a given market model is one of the major tasks of mathematical finance. The initial price $C(0)$ (resp. $P(0)$) is called the option **premium**.

To avoid trivial cases, assume that the strike price K satisfies

$$S(0)(1 + D) \leq K < S(0)(1 + U).$$

We now look for a portfolio (x, y) (of stocks and bonds) with the same value at time 1 as the option, the so-called **replicating** portfolio

$$V_{(x,y)}(1) = C(1).$$

This gives a pair of simultaneous equations

$$\begin{cases} xS(0)(1 + U) + yA(0)(1 + R) = S(0)(1 + U) - K, \\ xS(0)(1 + D) + yA(0)(1 + R) = 0, \end{cases}$$

which are easily solved

$$\begin{cases} x_C = \dfrac{S(0)(1 + U) - K}{S(0)(U - D)}, \\[2ex] y_C = -\dfrac{(1 + D)(S(0)(1 + U) - K)}{A(0)(U - D)(1 + R)}. \end{cases}$$

We claim that the price of the option should be the initial value of the replicating portfolio. This is based on the intuition that two portfolios identical at time 1 should have the same values at time 0. The justification involves the No Arbitrage Principle: in the case of different initial values, at time 1 we buy the cheaper and sell the more expensive asset and at time 1 we keep the profit. To do this requires trading in options, which leads us to extend the notion of the market.

We extend the notion of a portfolio by including the position in options: the extended **portfolio** is now a triple (x, y, z), where x represents the number of units of stock S, y the risk-free position (the money market account A), and z the number of units of the call option C held (z can be negative, which corresponds to taking a short position by writing and selling z options).

The definition of the **portfolio value** is extended in a natural way:

$$V_{(x,y,z)}(0) = xS(0) + yA(0) + zC(0),$$
$$V_{(x,y,z)}(1) = xS(1) + yA(1) + zC(1).$$

Remember that the second equation asserts equality of random variables! This allows us to formulate our key assumption in the same way as before.

Definition 2.6
We say that (x, y, z) is an arbitrage opportunity if $V_{(x,y,z)}(0) = 0$, $V_{(x,y,z)}(1) \geq 0$, and with positive probability $V_{(x,y,z)}(1) > 0$.

Assumption 2.7 No Arbitrage Principle
Arbitrage opportunities do not exist in the extended market.

In future models we shall always assume that the notion of portfolio is adjusted similarly to cover all securities in question.

Theorem 2.8
The No Arbitrage Principle implies that the price of the European call with strike price K is the value at time 0 of the replicating portfolio

$$C(0) = x_C S(0) + y_C A(0)$$

with x_C, y_C as above.

Proof Suppose $C(0) > x_C S(0) + y_C A(0)$. Then if we sell the expensive asset (option), and buy the cheap (portfolio (x_C, y_C)) we obtain the positive balance $C(0) - (x_C S(0) + y_C A(0))$. To construct an arbitrage we invest this amount in the money market account in addition to $y_C A(0)$ already held, so our portfolio is

$$(x, y, z) = (x_C, \frac{1}{A(0)}(C(0) - x_C S(0)), -1)$$

and its current value is zero. At time 1, in each scenario the value of this portfolio is

$$\begin{aligned} V_{(x,y,z)}(1) &= xS(1) + yA(1) + zC(1) \\ &= x_C S(1) + (C(0) - x_C S(0))(1 + R) - C(1) \\ &= (C(0) - (x_C S(0) + y_C A(0)))(1 + R) \end{aligned}$$

since by replication $C(1) = x_C S(1) + y_C A(0)(1 + R)$. But $C(0) - (x_C S(0) + y_C A(0)) > 0$ by our assumption, so we have a strictly positive outcome in each scenario at zero initial cost, more than is needed to contradict the No Arbitrage Principle.

If $C(0) < x_C S(0) + y_C$ we take the opposite position with the same result. □

Remark 2.9
The fact that the probabilities of the stock movements do not appear in the pricing formula is a paradox. One might expect that a call option where