HADAMARD EXPANSIONS AND HYPERASYMPTOTIC EVALUATION

An Extension of the Method of Steepest Descents

The author describes the recently developed theory of Hadamard expansions applied to the high-precision (hyperasymptotic) evaluation of Laplace and Laplace-type integrals. This new method builds on the well-known asymptotic method of steepest descents, of which the opening chapter gives a detailed account illustrated by a series of examples of increasing complexity. A discussion of uniformity problems associated with various coalescence phenomena, the Stokes phenomenon and hyperasymptotics of Laplace-type integrals follows. The remaining chapters deal with the Hadamard expansion of Laplace integrals, with and without saddle points. Problems of different types of saddle coalescence are also discussed. The text is illustrated with many numerical examples, which help the reader to understand the level of accuracy achievable. The author also considers applications to some important special functions.

This book is ideal for graduate students and researchers working in asymptotics.

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Contents

Preface

The aims of this book are twofold. The first is to present a detailed account of the classical method of steepest descents applied to the asymptotic evaluation of Laplace-type integrals containing a large parameter, and the second is to give a coherent account of the theory of Hadamard expansions. This latter topic, which has been developed during the past decade, extends the method of steepest descents and effectively 'exactifies' the procedure since, in theory, the Hadamard expansion of a Laplace or Laplace-type integral can produce unlimited accuracy.

Many texts deal with the method of steepest descents, some in more detail than others. The well-known books by Copson *Asymptotic Expansions* (1965), Olver *Asymptotics and Special Functions* (1997), Bleistein and Handelsman *Asymptotic Expansion of Integrals* (1975), Wong *Asymptotic Approximations of Integrals* (1989) and Bender and Orszag *Advanced Mathematical Methods for Scientists and Engineers* (1978) are all good examples. It is our aim in the first chapter to give a comprehensive account of the method of steepest descents accompanied by a set of illustrative examples of increasing complexity. We also consider the common causes of non-uniformity in the asymptotic expansions of Laplace-type integrals and conclude the first chapter with a discussion of the Stokes phenomenon and hyperasymptotics.

The next two chapters present the Hadamard expansion theory of Laplace and of Laplace-type integrals possessing saddle points. A study of these chapters makes it apparent how this theory builds upon and extends the method of steepest descents. Considerable emphasis is devoted to explaining the problems associated with coalescence phenomena, such as a saddle point coalescing either with another saddle point or with an endpoint of the integration interval. Methods for dealing with these difficulties in the Hadamard expansion procedure are carefully described. The monograph closes with sophisticated applications of the ideas developed in the earlier chapters to four particular special functions: the Bessel function $J_\nu(\nu x)$ of large order and argument, the Pearcey integral (a two-variable generalisation of the classical Airy function), the parabolic cylinder function $U(a, z)$ of large order and argument, and the logarithm of the gamma function.

In keeping with the last-mentioned text above, many of the examples in the later chapters are illustrated with numerical studies to better display the calibre of the asymptotic approximations obtained, a strategy that gives the non-expert practitioner a good sense of the method being showcased. This book should be accessible to anyone with a solid undergraduate background in functions of a single complex variable.

The author acknowledges the support of his institution, the University of Abertay, Dundee, which facilitated the writing of this book. A considerable debt of gratitude is owed to several colleagues who generously undertook a careful inspection of various sections and for their critical comments that have helped to improve the presentation of this text. The whole of Chapter 1 was read by N. M. Temme, with the first half of this chapter and Chapter 2 being read by T. M. Dunster; Chapters 2 and 3 were read by D. Kaminski, and C. J. Howls inspected the section on hyperasymptotics of Laplace-type integrals in Chapter 1. Finally, the non-specialist comments on the first part of Chapter 1 by J. S. Dagpunar were helpful. It is almost inevitable, however, that in spite of this careful examination some errors or misprints will have remained undetected, and the author requests the reader's forebearance for those that prove to be vexatious.