1

Asymptotics of Laplace-type integrals

In this opening chapter we present a detailed account, together with a series of examples of increasing complexity, of the classical method of steepest descents applied to Laplace-type integrals. Consideration is also given to the common causes of non-uniformity in the asymptotic expansions so produced due to a variety of coalescence phenomena. The chapter concludes with a brief discussion of the Stokes phenomenon and hyperasymptotics, both of which have undergone intense development during the past two decades. Such a preliminary discussion, as well as hopefully being of general interest in its own right, is necessary for the remaining chapters, since the Hadamard expansion procedure can be viewed as an 'exactification' of the method of steepest descents yielding hyperasymptotic levels of accuracy. Considerable space in the later chapters is devoted to showing how the Hadamard expansion procedure can be modified to deal with various coalescence problems.

1.1 Historical introduction

One of the most important methods of asymptotic evaluation of certain types of integral is known as the method of steepest descents. This method has its origins in the observation made by Laplace in connection with the estimation of an integral arising in probability theory of the form (Laplace, 1820; Gillespie, 1997)

$$I_n = \int_a^b f(x)\{g(x)\}^n dx = \int_a^b f(x)e^{n\psi(x)}dx \qquad (n \to +\infty).$$

Here $f(x)$ and $g(x)$ are real continuous functions defined on the interval $[a, b]$ (which may be infinite), with $g(x) > 0$ and $\psi(x) = \log g(x)$. Laplace argued that the dominant contribution to this integral as $n \to +\infty$ should arise from a neighbourhood of the point where $g(x)$ (or $\psi(x)$) attains its maximum value. In the simplest situation where $\psi(x)$ possesses a single maximum at the point $x = x_0 \in (a, b)$, so that $\psi'(x_0) = 0$, $\psi''(x_0) < 0$ and $f(x_0) \neq 0$, then $\psi(x)$ and $f(x)$ may be replaced by the leading terms in their Taylor series expansion. In a small neighbourhood of length δ either side of $x = x_0$, we then find

$$I_n \simeq \int_{x_0-\delta}^{x_0+\delta} f(x_0)e^{n\{\psi(x_0)+(x-x_0)^2\psi''(x_0)/2\}}dx$$
$$\simeq f(x_0)e^{n\psi(x_0)}\left(\frac{2}{-n\psi''(x_0)}\right)^{1/2}\int_{-u_*}^{u_*} e^{-u^2}du,$$

where $u_* = \delta(-n\psi''(x_0)/2)^{1/2}$. Assuming δ is chosen such that $n^{1/2}\delta \to \infty$ as $n \to +\infty$, we can replace the integration limits in the last integral over u by $\pm\infty$ and evaluate the integral, to obtain Laplace's result

$$I_n \simeq f(x_0)e^{n\psi(x_0)}\left(\frac{-2\pi}{n\psi''(x_0)}\right)^{1/2} \qquad (n \to +\infty).$$

This idea was subsequently employed by Cauchy (1829) in the estimation of the large-n behaviour of the coefficients a_n in certain series expansions, and in particular those in the Lagrange inversion series. This was motivated by the wish to determine the radius of convergence of such series expansions and to examine their behaviour on the circle of convergence. Let $g(z)$ denote a function that is analytic at the point $z = \alpha$ (with $g'(\alpha) \neq 0$) and $F(z)$ a function analytic at $z = 0$. Then the equation $z = wg(\alpha + z)$ has a unique solution $z = z(w)$ valid in a neighbourhood of $w = 0$ and $F(z(w)) = F(0) + \sum_{n=1}^{\infty} a_n w^n$. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i n}\oint_C F'(z)\frac{g^n(\alpha+z)}{z^n}\,dz = \frac{1}{2\pi i n}\oint_C F'(z)e^{n\psi(z)}dz,$$

where $\psi(z) = \log(g(\alpha + z)/z)$ and C is a circular contour surrounding $z = 0$. Cauchy then applied Laplace's argument in the complex plane (with $z = re^{i\theta}$): the circle C was expanded until it passed through the saddle point of $\psi(z)$ (given by the zeros of $\psi'(z) = 0$) closest to the origin. Cauchy never varied his choice of contour: he always took C to be a circle, thereby depriving his treatment of the necessary generality and incorrectly dealing with the case of multiple saddle points (Petrova and Solov'ev, 1997).

A quarter of a century later, Stokes (1850) investigated the asymptotics of the integral[1]

$$W(m) = \int_0^\infty \cos\tfrac{1}{2}\pi(w^3 - mw)\,dw \qquad (m \to +\infty)$$

in connection with the intensity of light in the neighbourhood of a caustic in the then new wave theory of light applied to the rainbow. The zeros of $W(m)$ corresponded to the location of darkbands in a system of supernumerary rainbows, of which up

[1] The integral $W(m)$ can be expressed in terms of the Airy function $\mathrm{Ai}(-\alpha m)$, where $\alpha = (\pi/2)^{2/3}3^{-1/3}$.

to 30 had been observed experimentally. By writing the cosine as the real part of its associated exponential and rotating the integration path through $-\pi/6$, Stokes reduced his integral to consideration of

$$\int_0^\infty \exp(-t^3+3q^2t)\,dt, \qquad q=(\pi/2)^{1/3}(m/3)^{1/2}e^{\pi i/6}.$$

He then proceeded to expand the integrand about $t=q$ (a saddle point) and took his integration path in the neighbourhood of this point in the direction in which the imaginary part of the exponent in the exponential factor remained constant (the path of steepest descent through $t=q$, see §1.2.1). By bounding the contribution from different parts of the deformed path, he was able to establish that the dominant contribution to this integral arose from a neighbourhood of $t=q$. Stokes thereby obtained the result

$$W(m)\sim(2/3)^{1/2}(m/3)^{-1/4}\cos\{\pi(m/3)^{3/2}-\pi/4\} \quad (m\to+\infty),$$

from which he was able to calculate approximations to the first 50 zeros of $W(m)$. Although Stokes did not mention the terms saddle point or path of steepest descent, he was, nevertheless, effectively employing the ideas of the saddle-point method in the complex plane. A more detailed account of this problem together with a discussion of Stokes' other mathematical contributions can be found in Paris (1996).

In 1863, Riemann investigated an asymptotic approximation for the Gauss hypergeometric function ${}_2F_1(n-c, n+a+1; 2n+a+b+2; x)$ when $n\to+\infty$ and the variable x has complex values. This function can be expressed in terms of the integral

$$I_n=\int_0^1 f(t)\left(\frac{t(1-t)}{1-xt}\right)^n dt=\int_0^1 f(t)e^{n\psi(t)}dt,$$

where $f(t)=t^a(1-t)^b(1-xt)^c$ and $\psi(t)=\log\{t(1-t)/(1-xt)\}$. This integral defines an analytic function in the complex x-plane cut along the real axis from $x=1$ to $x=+\infty$. His paper (Riemann, 1863) on this calculation was never finished in his lifetime: the second part, containing the asymptotic calculation, contained only key expressions to guide him during the writing-up process. The text, together with some of the computations in this posthumous paper, were filled in by H. Schwarz. Riemann determined the two saddle points t_{s1} and t_{s2} of $\psi(t)$ (given by $\psi'(t)=0$) as

$$t_{s1}=\frac{1}{1+\sqrt{1-x}}, \qquad t_{s2}=\frac{1}{1-\sqrt{1-x}},$$

with the branch of the square root having positive real part in the cut x-plane. He then proceeded to show that the integration path could be deformed to pass through the saddle t_{s1} and argued that as $n\to+\infty$ the dominant contribution to I_n arises from a neighbourhood of t_{s1}. By integrating along the direction of steepest descent through the saddle, he obtained the result

$$I_n \sim \frac{(\pi/n)^{1/2}(1-x)^{(b+c)/2+\frac{1}{4}}}{(1+\sqrt{1-x})^{2n+a+b+1}} \qquad (n \to +\infty),$$

although no mention was made of the sector of validity (in the complex x-plane) of this approximation.

This paper is frequently cited as containing the germ of the idea of the method of steepest descents. If Riemann had used the whole path connecting $t = 0$ to $t = 1$, he would have obtained the full asymptotic expansion of I_n, instead of just the leading term. It is clear, however, that he must have been in possession of the steepest descent technique since he used this method with great skill in his famous expansion[2] of the function $Z(t)$, related to the Riemann zeta function $\zeta(s)$ on the critical line $s = \frac{1}{2}+it$, as $t \to +\infty$; for a detailed account, see Edwards (1974, Ch. 7).

An interesting survey paper by Petrova and Solov'ev (1997) discusses these developments (with the exception of those of Stokes) in greater detail. These authors also point out the work of the Russian mathematician Nekrosov who, about 20 years after Riemann, considered Cauchy's problem of the determination of the leading behaviour of the coefficients in the Langrange inversion series. He considered the problem in general and discussed the situation when there are several saddle points of arbitrary multiplicity. He showed the existence of a closed contour passing through the saddle points along the directions of steepest descent, but only obtained the dominant contribution from each saddle.

Finally, the first person to obtain a full asymptotic expansion in a specific case was the physicist Debye (Debye, 1909). He developed the method, after seeing Riemann's paper, for the Hankel functions defined in the form

$$H_\nu^{(1,2)}(x) = -\frac{1}{\pi}\int_{-\infty i}^{\infty i \mp \pi} e^{-ix\psi(t)}dt, \quad \psi(t) = \sin t - \alpha t, \quad \alpha = \frac{\nu}{x}$$

for large positive values of x and the order ν. The integrand has two saddle points at which $\psi'(t) = 0$ in the strip $-\pi < \mathrm{Re}(t) < \pi$, which are situated symmetrically about the origin on the real axis when $0 < \alpha < 1$ and on the imaginary axis when $\alpha > 1$. Debye deformed the contours into steepest descent paths passing through one or both saddles, and then converted the integrals leading up to and away from a saddle into a Laplace integral of the form $\int_0^\infty e^{-xu}\phi(u)\,du$ by an appropriate change of variable. Expansion of $\phi(u)$ about $u = 0$ into a series of fractional powers of u then enabled him to integrate term by term to obtain the asymptotic expansion of $H_\nu^{(1,2)}(x)$ for x and $\nu \to +\infty$.

Debye also considered the situation when the variable x and ν are large and nearly equal. In this case, the two saddles in the strip $-\pi < \mathrm{Re}(t) < \pi$ coalesce to form a double saddle point at the origin when $x = \nu$. A slight modification of his argument

[2] This is the Riemann–Siegel formula which was discovered by Siegel in Riemann's papers and published in 1932.

then enabled him to derive expansions for the Hankel functions as x and $\nu \to +\infty$ when $x \simeq \nu$.

1.2 The method of steepest descents

1.2.1 Preliminaries

The type of integral under consideration is

$$I(\lambda) = \int_C e^{\lambda\psi(t)} f(t)\, dt, \tag{1.2.1}$$

where λ is a large positive[3] parameter and C is a path of finite or infinite extent in the complex t-plane which is chosen such that $I(\lambda)$ converges. The amplitude function $f(t)$ and phase function $\psi(t)$ are assumed to be analytic on and near the path C and, for the purpose of this section, to be independent of λ.

Let $\psi(t) = U(x, y) + iV(x, y)$ where $t = x + iy$ and U, V, x, y are real. When λ is large a small displacement along the path C causing a small change in $V(x, y)$ will, in general, produce a rapid oscillation of the sinusoidal terms in $\exp(\lambda\psi(t))$. This has the consequence that the contribution to the integral will be subject to considerable cancellation between neighbouring parts of the path. An obvious remedy is to choose a path on which $V(x, y)$ is constant, thereby removing the rapid oscillations of the integrand. The most rapidly varying part of the integrand will then be $\exp(\lambda U)$ and the dominant contribution will arise from a neighbourhood of the point where $U(x, y)$ is greatest.

The choice of a path with $V(x, y) =$ constant has another major advantage. For, such paths are those on which $U(x, y)$ *changes the most rapidly*. To see this, let us consider a small displacement from the point t_0 given by $t = t_0 + se^{i\theta}$, where $s > 0$ and θ is a phase angle. Then the rate of change of $U(x, y)$ is given by

$$\frac{dU}{ds} = \frac{\partial U}{\partial x}\cos\theta + \frac{\partial U}{\partial y}\sin\theta = U_x \cos\theta + U_y \sin\theta.$$

Regarded as a function of θ, dU/ds will have a stationary point when its derivative D with respect to θ vanishes; that is, when $D := -U_x \sin\theta + U_y \cos\theta = 0$. Since $\psi(t)$ is an analytic function of the complex variable t in the neighbourhood of C, its derivatives are constrained to satisfy the Cauchy–Riemann equations

$$U_x = V_y, \qquad U_y = -V_x \tag{1.2.2}$$

in this neighbourhood. Substitution of these equations into the above stationary condition $D = 0$ yields

$$V_x \cos\theta + V_y \sin\theta = 0.$$

[3] If $\lambda = |\lambda|e^{i\phi}$ is complex, then we can take $|\lambda|$ as the large parameter and absorb the phase term $e^{i\phi}$ into $\psi(t)$.

But this last equation states that $dV/ds = 0$. Since

$$\left|\frac{dU}{ds}\right|^2 = |U_x \cos\theta + U_y \sin\theta|^2 = U_x^2 + U_y^2 - (U_x \sin\theta - U_y \cos\theta)^2$$
$$= U_x^2 + U_y^2 - D^2,$$

it is seen that the stationary direction ($D = 0$) corresponds to a maximum in the absolute value of the rate of change of $U(x, y)$ with respect to s. *Thus a path* $V(x, y) =$ constant *is one along which* $U(x, y)$ *changes the most rapidly.*

To obtain a geometrical insight into the nature of such paths we recall some well-known properties of functions of a complex variable. Since $|e^{\lambda\psi(t)}| = e^{\lambda U}$, we are interested in the modular surface S defined by $U = U(x, y)$, where the U-axis is perpendicular to the x, y-plane. A point (x_0, y_0) on this surface where $U_x(x_0, y_0) = U_y(x_0, y_0) = 0$ is a stationary point. From (1.2.2) we have $U_{xx} = V_{yx}$ and $U_{yy} = -V_{xy}$, so that

$$U_{xx} + U_{yy} = 0$$

and $U(x, y)$ is a harmonic function. Then, since the quantity

$$U_{xx}U_{yy} - U_{xy}^2 = -U_{xx}^2 - U_{xy}^2 < 0,$$

the stationary point (x_0, y_0) must be a saddle point. Hence all stationary points on S can only be saddle points (or cols); the surface S has no maxima[4] and no minima (except for isolated zeros of $\psi(t)$). Application of the Cauchy–Riemann equations again shows that $\psi'(t) = U_x(x, y) + iV_x(x, y) = U_x(x, y) - iU_y(x, y)$, so that the stationary point (x_0, y_0) must be a saddle point of the phase function $\psi(t)$.

The shape of the modular surface S can also be visualised on the x, y-plane by constructing the level curves on which $U(x, y) =$ constant. From (1.2.2) it follows that

$$U_x V_x + U_y V_y = \nabla U \cdot \nabla V = 0,$$

where $\nabla \equiv \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y$ is the two-dimensional gradient operator. Thus the families of curves corresponding to constant values of $U(x, y)$ and $V(x, y)$ are orthogonal at all their points of intersection. The regions where $U(x, y) > U(x_0, y_0)$ are called *hills* (or *ridges*) and those where $U(x, y) < U(x_0, y_0)$ are called *valleys*. The level curve through the saddle, $U(x, y) = U(x_0, y_0)$, separates the immediate neighbourhood of the saddle point (x_0, y_0) into a series of hills and valleys.

To see this topography, let us suppose that $t_s = x_0 + iy_0$ is a saddle point of order $m - 1$ (with $m \geq 2$); that is the first $m - 1$ derivatives of $\psi(t)$ at t_s all vanish

$$\psi'(t_s) = \psi''(t_s) = \cdots = \psi^{(m-1)}(t_s) = 0,$$

[4] This can also be seen by application of the maximum-modulus principle.

with $\psi^{(m)}(t_s) = Ae^{i\phi}$, $A > 0$. Then, if $t = t_s + re^{i\theta}$, $r > 0$, we have

$$\psi(t) - \psi(t_s) = (Ar^m/m!)e^{i(m\theta+\phi)} + \cdots$$

and hence

$$\begin{pmatrix} U(x,y) - U(x_0,y_0) \\ V(x,y) - V(x_0,y_0) \end{pmatrix} = \frac{Ar^m}{m!}\begin{pmatrix} \cos \\ \sin \end{pmatrix}(m\theta+\phi) + \cdots$$

in a neighbourhood of t_s. The directions of the paths of constant $U(x, y)$ and $V(x, y)$ are consequently determined by setting the right-hand side of the above expression equal to zero to find

$$\theta = (k + \tfrac{1}{2}\delta)\frac{\pi}{m} - \frac{\phi}{m}, \quad \delta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (k = 0, 1, \ldots, 2m-1), \tag{1.2.3}$$

respectively. Therefore, there are $2m$ equally spaced steepest directions from $t_s : m$ directions of steepest descent and m directions of steepest ascent. In the neighbourhood of t_s, the level curves $U = U(x_0, y_0)$ form the boundaries of m valleys surrounding the saddle point, in which $\cos(m\theta + \phi) < 0$, and m hills on which $\cos(m\theta + \phi) > 0$. The valleys and hills are situated respectively entirely below and above the saddle point, and each has angular width equal to π/m.

The steepest paths satisfy $\sin(m\theta+\phi) = 0$, and so have directions given by (1.2.3) with $\delta = 0$. The directions of steepest descent at the saddle point t_s are therefore

$$\theta = (2k+1)\frac{\pi}{m} - \frac{\phi}{m} \qquad (k = 0, 1, \ldots, m-1;\ m \geq 2). \tag{1.2.4}$$

The topography of the surface S near the saddle point is shown in Fig. 1.1 when $\phi = \frac{1}{2}\pi$ for the cases $m = 2$ (which corresponds to the most commonly occurring situation of a first-order saddle) and $m = 3$. We remark that, away from the immediate neighbourhood of the saddle point, the projection of the steepest paths and the valley boundaries onto the t-plane will, in general, be curved paths.

A typical situation has the contour of integration C in (1.2.1) beginning and ending at infinity in the valleys (which is necessary for convergence). The contour is then deformed as far as possible into paths of steepest descent running along the bottoms of valleys and crossing over from one valley to the next over a saddle point; see Fig. 1.2. An interesting analogy has been given by De Bruijn (1958, p. 80) in the form of a person wishing to travel between two points in a mountainous region: if the two points are in different valleys, then the least effort should involve a passage via a col. A similar idea is presented in Greene and Knuth (1982, §4.3.3) in relation to a lazy hiker who will choose a path that crosses a ridge at its lowest point; but unlike the truly lazy hiker, who would probably choose a zig-zag path, the best path takes the steepest ascent to the col.

We remark that the determination of the paths of steepest descent in particular cases can be quite difficult. It is usually a simple matter to locate the saddle points of a given phase function $\psi(t)$ and the directions of steepest descent away from these

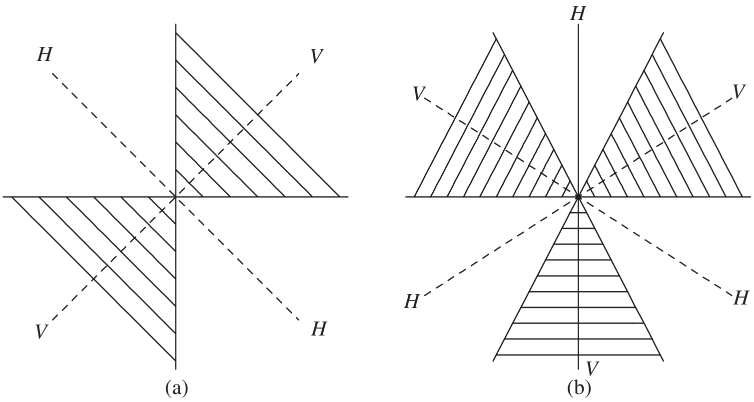

Figure 1.1 Paths of steepest descent and ascent (dashed lines) in the neighbourhood of the saddle point t_s when $\phi = \frac{1}{2}\pi$ and (a) $m = 2$ and (b) $m = 3$. The shaded regions denote the valleys (V) and the unshaded regions denote the hills (H).

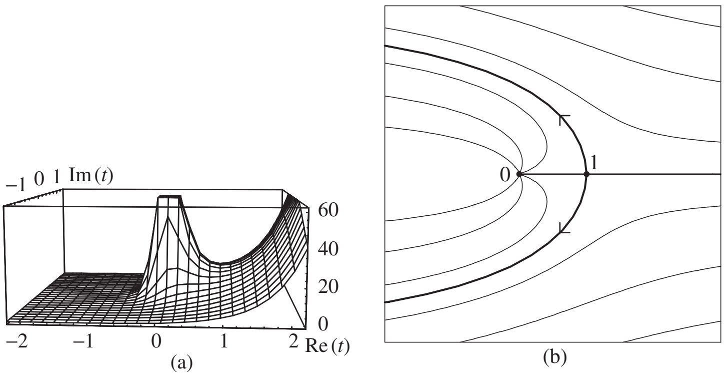

Figure 1.2 (a) The modular surface of $F = t^{-3}e^{3t}$ possessing a saddle point of order 1 ($m = 2$) at $t = 1$ and (b) the associated paths of constant Im(F). The heavy line denotes the steepest descent path through the saddle and the arrows denote the direction of descent.

saddles. It is also easy to determine the valleys at infinity. What is not so straightforward is how the various steepest descent paths connect up with the valleys. Often an intelligent guess is successful, especially when the variable λ is real. However, *Mathematica* can be employed to great advantage in the construction of the paths of steepest descent. This is the approach we adopt in all the examples in this book.

On a steepest path through a saddle point t_s we have

$$\psi(t) = \psi(t_s) - u,$$

where u is real. Unless this path[5] connects with another saddle or a singularity of $\psi(t)$, the variable u either increases monotonically to $+\infty$ along a steepest descent

[5] A path of steepest descent terminates only at infinity or at singular points of $\psi(t)$.

path or decreases monotonically to $-\infty$ along a steepest ascent path. In general, the contour will consist of a series of steepest descent paths, each running from a saddle point down a valley out to infinity or to a singularity of the phase function $\psi(t)$. This leads to finding the asymptotic expansion of integrals of the type

$$e^{\lambda\psi(t_s)}\int_{t_s}^{T} e^{-\lambda u} f(t)\,dt,$$

where T denotes some point on a steepest descent path, and adding the contributions from each relevant saddle point. The most common situation has $T = +\infty$, although T can be finite if the integration path encounters another saddle or, of course, if the original path C in (1.2.1) is finite.

1.2.2 Asymptotic expansion of $I(\lambda)$

In this section we determine the asymptotic expansion of the integral

$$I(\lambda) = \int_C e^{\lambda\psi(t)} f(t)\,dt \qquad (1.2.5)$$

for $\lambda \to +\infty$, where the path C is a steepest descent path that commences at a saddle point t_s of order $m-1$. The derivation is formal but the expansion process is justified by a useful result known as Watson's lemma, which we state and prove in §1.2.4.

We commence by giving the definition of an asymptotic expansion.

Definition 1.1 Let $f(z)$ be a function of a real or complex variable z, $\sum c_k z^{-k}$ a formal power series (convergent or divergent) and $R_N(z)$ the difference between $f(z)$ and the Nth partial sum of the series; that is

$$f(z) = \sum_{k=0}^{N-1} c_k z^{-k} + R_N(z).$$

Then, if for each fixed value of N

$$R_N(z) = O(z^{-N})$$

as $z \to \infty$ in a certain unbounded region **R**, we say that the series $\sum c_k z^{-k}$ is an asymptotic expansion of $f(z)$ and write

$$f(z) \sim \sum_{k=0}^{\infty} c_k z^{-k} \qquad (z \to \infty \text{ in } \mathbf{R}). \qquad (1.2.6)$$

This definition is due to Poincaré (1886). The formal series so obtained is also referred to as an asymptotic expansion of *Poincaré type*, or an asymptotic expansion in the *Poincaré sense*, or more simply as a *Poincaré expansion*.

In the integral $I(\lambda)$ in (1.2.5) we put

$$\psi(t) = \psi(t_s) - u, \qquad (1.2.7)$$

where u is non-negative and monotonically increasing as one progresses down the steepest descent path. This produces

$$I(\lambda) = e^{\lambda\psi(t_s)} \int_{t_s}^{T} e^{-\lambda u} f(t)\, dt = e^{\lambda\psi(t_s)} \int_0^{T'} e^{-\lambda u} f(t) \frac{dt}{du}\, du, \qquad (1.2.8)$$

where $T' > 0$ is the map of T in the u-plane. For large positive λ, the exponential factor $e^{-\lambda u}$ in (1.2.8) decays rapidly so that the main contribution comes from the neighbourhood of $u = 0$. Accordingly, to determine the asymptotic expansion of $I(\lambda)$ for $\lambda \to +\infty$, we require the series expansion of $f(t)dt/du$ in ascending powers of u. This expansion is substituted into the integral (1.2.8) which is then integrated term by term.

If t_s is a saddle point of order $m-1$, then

$$\psi(t) = \psi(t_s) - \sum_{k=0}^{\infty} a_k (t - t_s)^{m+k} \qquad (1.2.9)$$

valid in some disc surrounding t_s, where $a_k = -\psi^{(m+k)}(t_s)/(m+k)!$ and $a_0 \neq 0$. Comparison of this expansion with (1.2.7) shows that

$$u = \sum_{k=0}^{\infty} a_k (t - t_s)^{m+k}. \qquad (1.2.10)$$

If we let $u = \tau^m$, then for small $|t - t_s|$

$$\tau = a_0^{1/m}(t - t_s)\left\{1 + \frac{a_1}{ma_0}(t - t_s) + \frac{1}{m}\left(\frac{a_2}{a_0} - \frac{(m-1)a_1^2}{2ma_0^2}\right)(t - t_s)^2 + \cdots\right\},$$

where $a_0^{1/m}$ takes its principal value. It follows that τ is a single-valued analytic function of t in the neighbourhood of t_s and that $\tau'(t_s) \neq 0$. By the inverse function theorem (Copson, 1935, p. 121; Jeffreys and Jeffreys, 1972, p. 380) we then have

$$t - t_s = \sum_{k=1}^{\infty} \alpha_k \tau^k = \sum_{k=1}^{\infty} \alpha_k u^{k/m}, \qquad (1.2.11)$$

where

$$\alpha_1 = \frac{1}{a_0^{1/m}}, \qquad \alpha_2 = -\frac{a_1}{ma_0^{1+2/m}}, \qquad \alpha_3 = \frac{(m+3)a_1^2 - 2ma_0a_2}{2m^2a_0^{2+3/m}}, \ldots .$$

This gives one inversion of (1.2.10); the others are obtained by replacing u in (1.2.11) by $ue^{2\pi in}$, with n an integer satisfying $1 \leq n \leq m-1$.

In simple cases it is possible to determine the coefficients α_k in closed form by application of the Lagrange inversion theorem, but this is not practicable in more complicated cases. This important theorem can be stated as follows (Copson, 1935, p. 125; Whittaker and Watson, 1952, p. 133; Jeffreys and Jeffreys, 1972, p. 383).