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Fixed-income instruments

At its simplest, an interest rate is the rate that is charged or paid for the use of money. It is often expressed as an annual percentage of the notional amount. Throughout the text we will generally focus on what are known as 'interbank rates'. These are the interest rates at which banks borrow from and lend to each other in the interbank, or over-the-counter (OTC) market. The most important example of an interbank rate is the London Interbank Offered Rate, or LIBOR. The LIBOR rate is the interest rate at which banks offer to lend unsecured funds to each other in the London wholesale money market. Another related market rate is the swap rate, which is the fixed rate that a bank is willing to exchange for a series of payments based on the LIBOR rate.

In this chapter we present some basic terminology and definitions, together with an overview of fixed-income instruments such as forward rate agreements (FRAs), swaps, floating-rate notes and fixed-coupon bonds. Government-backed securities form another important class of interest rate instruments. In the USD market, securities such as Treasury bonds, Treasury notes and Treasury bills are issued by the US Treasury to finance government debt. All instruments are assumed to be default free, that is, we do not account for the possibility that the issuers may fail to honour their commitments.

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We begin with the definition of the zero-coupon bond. Such bonds are not actively traded within the interbank market. The reason they are important, however, is because interbank interest rates such as LIBOR and swap rates can be defined in terms of zero-coupon bonds. The set of zero-coupon bonds for various time horizons is known as the zero-coupon curve. How the zero-coupon curve is estimated from market data such as LIBOR and swap rates is also discussed.

In this chapter, and indeed throughout this volume, time is measured in years.

1.1 Interest rates and bonds

A **zero-coupon bond** or **discount bond** with maturity date T is a financial contract that guarantees the holder one dollar at time T. The bond can be thought of as the value at time $t < T$ of one dollar to be paid at time T. The zero-coupon bond maturing at time T is often referred to as a T-bond, and its price at time t is denoted by $B(t, T)$. Therefore, a zero-coupon bond is parameterised by two time indices, the current time t and the maturity date T. By definition, $B(T, T) = 1$, and we have

$$0 < B(t, T) < 1$$

for $t < T$.

The dependence of $B(t, T)$ on the maturity date T is known as the **term structure of discount factors** or **zero-coupon curve** at time t. The curve is a decreasing function of maturity.

Spot interest rates

Having defined the zero-coupon bond $B(t, T)$, we now introduce the notion of the simply compounded interest rate. The **simply compounded spot rate** at time t for maturity T is defined as the annualised rate of return from holding the bond from time t until maturity T. It is denoted by $L(t, T)$, and is defined as

$$L(t, T) = \frac{1 - B(t, T)}{(T - t)B(t, T)}. \tag{1.1}$$

The bond price $B(t, T)$ can be expressed in terms of the spot rate $L(t, T)$ as

$$B(t, T) = \frac{1}{1 + (T - t)L(t, T)}. \tag{1.2}$$

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If interest rates are positive, we must have

$$B(t,S) > B(t,T)$$

for $t \leq S < T$.

An important example of a simply compounded rate is the **London Interbank Offered Rate** (**LIBOR**). This is the interest rate at which banks offer to lend unsecured funds to each other in the London wholesale money market. From now on we shall identify $L(t,T)$ with the LIBOR rate.

Remark 1.1
LIBOR is the primary benchmark for short-term interest rates. Daily fixings of LIBOR are published by the British Bankers Association (BBA) shortly after 11 a.m. (GMT). It is a filtered average of quotes provided by a number of banks and can be thought of as representing the lowest real-world cost of unsecured funding in the London money market. It is produced for ten major currencies, Pound Sterling, US Dollar, Euro, Japanese Yen, Swiss Franc, Canadian Dollar, Australian Dollar, Swedish Krona, Danish Krona and the New Zealand Dollar. Fifteen maturities are quoted for each currency ranging from overnight to 12 months. LIBOR rates are widely used as a reference rate for a range of vanilla financial instruments such as forward rate agreements, short-term interest rate futures contracts and interest rate swaps.

The Euro Interbank Offered Rate (EURIBOR) is another example of a money-market rate and is compiled by the European Banking Federation. The EURIBOR rate is the benchmark rate for EUR-denominated instruments.

Remark 1.2
We defined the simply compounded spot rate at time t as the rate of return over the interval $[t,T]$; see (1.1). For the case of USD LIBOR, however, the accrual period starts two London business days after time t (the date on which the rate becomes fixed). For example, for the six-month USD LIBOR spot rate on 15 March 2010 the accrual period begins on 17 March 2010 and ends on 17 September 2010. These timing conventions differ from currency to currency. Interest is calculated on an Actual/360 basis (see Remark 1.6). Throughout the text we will assume that spot rates fix and start on the same date, unless explicitly stated otherwise.

Interest rates quoted in the market are almost always simply compounded. However, it can be mathematically more convenient to work with continuously compounded rates. The **continuously compounded spot rate** is the

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annualised logarithmic rate of return from holding the bond from time t until maturity T. It is denoted by $R(t,T)$, and is defined as

$$R(t,T) = -\frac{\ln B(t,T)}{T-t}.$$

The zero-coupon bond price can be expressed in terms of $R(t,T)$ as

$$B(t,T) = e^{-R(t,T)(T-t)}. \tag{1.3}$$

The continuously compounded spot rate can be thought of as a measure of the implied interest rate offered by the bond and is sometimes referred to as the **yield to maturity**. The graph of $R(t,T)$ versus maturity T is known as the **yield curve** (see Figures 1.1 and 1.2). Yield curves are typically increasing or decreasing functions of T, but can often be inverted or 'hump' shaped.

Exercise 1.1 Consider an annually compounded spot rate $L(0,T)$ maturing in one year, i.e. $T = 1$. Compute the continuously compounded spot rate $R(0,T)$ when $L(0,T) = 5\%$.

Time value of money

The above definitions express the principle that today's value of one dollar paid at some time in the future is less than one dollar paid today. This is known as the **time value of money**.

A closely related notion is **discounted value** or **present value** (**PV**). It is the value today of a deterministic (known in advance) future payment or a series of deterministic future payments. We use the discount bond to express the present value. For example, an amount A known at time t to be paid at time $T > t$ has present value $B(t,T)A$ at time t.

The present value of a deterministic payment should not be confused with the more general concept of the discounted value of a random future payment. If we have a random payment X at some future time $T > t$, its discounted value $B(t,T)X$ at time t is also a random variable, whose value may be unknown at time t.

Exercise 1.2 Consider a perpetual bond that pays one dollar at the end of each year forever. Assuming that $B(0,T) = (1+r)^{-T}$, where r is a constant annually compounded rate of interest, show that the present

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value of the perpetual bond, that is, the sum of the present values of all the payments, can be written as a geometric series. Simplify the series to find the present value of the perpetual bond for $r = 5\%$.

The bond price is a stochastic process

If the bond price were deterministic, then the following would have to be true.

Proposition 1.3
Let $t < S < T$. If the zero-coupon bond price $B(S,T)$ were known at time t (i.e. deterministic), then in the absence of arbitrage we would have

$$B(t,T) = B(t,S)B(S,T). \tag{1.4}$$

Proof Suppose that $B(t,T) < B(t,S)B(S,T)$. Consider this strategy.

- At time t we buy (go long) a T-bond and sell (go short) an amount $B(S,T)$ of S-bonds to give an income of $B(t,S)B(S,T) - B(t,T) > 0$.
- At time S our short position in S-bonds matures and we are required to pay the amount $B(S,T)$. We raise this amount by selling one T-bond.
- At time T our net position will be zero. The long position in T-bonds purchased at time t cancels the short position in T-bonds purchased at time S.

Our strategy created a risk-free profit of $B(t,S)B(S,T) - B(t,T) > 0$ at time t, violating the no-arbitrage assumption. By adopting the opposite strategy, we can see that the reverse inequality $B(t,T) > B(t,S)B(S,T)$ would also give rise to an arbitrage opportunity. □

If we were to perform an empirical analysis of a bond price time series, it would quickly become apparent that condition (1.4) is not satisfied. The zero-coupon bond price should therefore be modelled as a stochastic process that evolves towards a known value at time T.

1.2 Forward rate agreements

Let $t < S < T$. A **forward rate agreement** (**FRA**) is a contract entered into at time t, when the issuer agrees to pay the holder at time T the LIBOR rate $L(S,T)$ in exchange for a fixed rate K applied to a notional amount N. The value of the payoff at time T is given by

$$\tau(K - L(S,T))N, \tag{1.5}$$

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where $\tau = T - S$ is the accrual period. Without loss of generality, we can assume a unit notional, $N = 1$. By definition, it costs nothing to enter into a FRA. Taking the time t value of the cash flows described above and setting the resulting sum equal to zero, we can find the value of the fixed rate K such that the FRA is zero. The time t value of the fixed interest payment τK is simply the discounted value $\tau K B(t, T)$. The time t value of the floating payment is given by the following result.

Proposition 1.4

The arbitrage-free value at time t of the LIBOR-based payment $\tau L(S, T)$ at time T is $B(t, S) - B(t, T)$.

Proof To see this consider the following strategy.

- At time t we buy (go long) an S-bond and sell (go short) a T-bond.
- At time S the long position in S-bonds matures to yield one dollar. Use this income to buy an amount $1/B(S, T)$ of T-bonds.
- At time T our net position will be $\frac{1}{B(S,T)} - 1$, which is equal to $\tau L(S, T)$.

We have replicated the payment at time T using a self-financing strategy with an initial cost of $B(t, S) - B(t, T)$. In the absence of arbitrage this must be the value of the floating payment at time t. □

The value of the FRA at time t is therefore

$$B(t, T)\tau K - B(t, S) + B(t, T).$$

Setting this equal to zero and solving for K, we find that the value of the fixed rate, known as the **forward LIBOR rate** or simply the **forward rate** and denoted by $F(t; S, T)$, is

$$F(t; S, T) = \frac{B(t, S) - B(t, T)}{\tau B(t, T)}. \tag{1.6}$$

The forward rate is a simply compounded rate parameterised by three time arguments: the present time t, the start of the spot LIBOR rate $S > t$ and the maturity date $T > S$.

Exercise 1.3 Consider two annually compounded spot rates $L(0, S)$ and $L(0, T)$, maturing in one and two years respectively, i.e. $S = 1$ and $T = 2$. Compute $L(0, T)$ when $L(0, S) = 4\%$ and the forward rate $F(0; S, T) = 5.5\%$.

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Exercise 1.4 For two annually compounded spot rates $L(0,1) = 4\%$ and $L(0,2) = 5\%$, maturing in one and two years respectively, compute the one-year to two-year forward rate $F(0;1,2)$.

1.3 Forward interest rates and forward bond price

Assume we wish to enter into an agreement at time t to purchase a T-bond at time S, where $t < S < T$. One of the simplest ways of determining the correct (arbitrage-free) amount A known at time t that we need to pay at time S is to take the time t value of the cash flows and set the resulting sum equal to zero,

$$-AB(t,S) + B(t,T) = 0.$$

The arbitrage-free amount we need to pay at time S to purchase the T-bond is known as the **forward bond price** or **forward discount factor**. It is denoted by $\mathbf{FP}(t;S,T)$ and given by

$$\mathbf{FP}(t;S,T) = \frac{B(t,T)}{B(t,S)}. \tag{1.7}$$

By rearranging (1.6), it can be seen that the forward bond price can be expressed in terms of the forward rate as

$$\mathbf{FP}(t;S,T) = \frac{B(t,T)}{B(t,S)} = \frac{1}{1+(T-S)F(t;S,T)}.$$

Therefore, we can think of the forward rate as the simply compounded rate of return over the time interval $[S,T]$ implied by the forward bond price.

We can quote forward rates as either simple rates or continuously compounded rates. The **continuously compounded forward rate** at time t for expiry S and maturity T is denoted by $R(t;S,T)$. It is found by solving

$$\frac{B(t,T)}{B(t,S)} = \mathrm{e}^{-R(t;S,T)(T-S)}. \tag{1.8}$$

The above can be written as

$$R(t;S,T) = -\frac{\ln B(t,T) - \ln B(t,S)}{T-S}.$$

Using (1.3), we can write this in terms of the continuously compounded

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spot rates as

$$R(t; S, T) = \frac{R(t, T)(T - t) - R(t, S)(S - t)}{T - S}.$$

Exercise 1.5 Given a continuously compounded spot rate $R(0, 1) = 4\%$ and a continuously compounded forward rate $R(0, 1, 2) = 5.5\%$, compute the spot rate $R(0, 2)$.

Remark 1.5

The forward rate $F(t; S, T)$ could be taken as a predictor of the actual spot interest rate $L(S, T)$ at time S. Indeed, $F(t; S, T)$ is the expectation of $L(S, T)$ under what is known as the T-forward measure (see Section 2.4 for more details).

Instantaneous rates

The **instantaneous forward rate** at time t for maturity T, which we denote by $f(t, T)$, can be thought of as the rate of return over an infinitesimally small time interval $[T, T + \delta T]$ or, more precisely,

$$f(t, T) = \lim_{\delta T \to 0} R(t, T, T + \delta T) = -\frac{\partial \ln B(t, T)}{\partial T}. \tag{1.9}$$

The dependence of $f(t, T)$ on the maturity T is known as the **term structure of forward rates** (or **forward curve**) at time t.

A related concept is the **instantaneous short rate** or **risk-free rate** at time t, denoted by $r(t)$. It is the rate of return over the infinitesimal time interval $[t, t + \delta t]$, and is defined in terms of the instantaneous forward rate as

$$r(t) = f(t, t).$$

Although they are abstract concepts, the instantaneous forward and short rates play an important role in stochastic interest rate modelling. In Chapter 3 we cover models based on the short rate and then in Chapter 4 we study the seminal Heath–Jarrow–Morton model of the dynamics of the term structure of forward rates.

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Bond price formula

Integrating (1.9) over the time interval $[t, T]$, we can see that

$$\int_t^T f(t,u)du = -\ln B(t,u)\Big|_t^T = -\ln B(t,T). \quad (1.10)$$

Hence the zero-coupon bond price can be expressed in terms of the instantaneous forward rates as

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right). \quad (1.11)$$

Exercise 1.6 Show that the instantaneous forward rate $f(t,T)$ and the continuously compounded spot rate $R(t,T)$ are related by

$$f(t,T) = R(t,T) + (T-t)\frac{\partial}{\partial T}R(t,T).$$

Remark 1.6

In the expression for the simply compounded forward rate the interest accrues over the time interval $[S, T]$. In reality, however, how interest accrues over time is determined by **day-count conventions**. There are three main cases.

- Actual/365: assume there are 365 days in a year, and calculate the actual number of days between the dates and divide by 365.
- 30/360: assume there are 30 days in a month and 360 days in a year, and calculate the number of days accordingly and divide by 360.
- Actual/360: assume there are 360 days in a year, and calculate the actual number of days between the dates and divide by 360.

For the market LIBOR rates the accrual basis is Actual/360.

Example 1.7

Consider a three-month USD LIBOR rate beginning on 17 March 2010 and maturing three months later on 17 June 2010. The accrual period is $92/360 = 0.255\,55$.

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1.4 Money market account

The **money market account** is a risk-free security where interest accrues continuously at the instantaneous short rate $r(t)$. The short rate is typically modelled as a stochastic process with the assumption that almost all sample paths are Lebesgue integrable. The value of the money market account at time t is denoted by $B(t)$, and is defined by the differential equation

$$dB(t) = r(t)B(t)dt$$

with $B(0) = 1$. Solving, we have

$$B(t) = \exp\left(\int_0^t r(u)du\right).$$

The money market account can be thought of as the amount earned by starting with a unit amount at time 0 and continually reinvesting it at the short rate $r(t)$ over the infinitesimal time interval $[t, t + \delta t]$. The money market account is often referred to as the bank-account numeraire.

1.5 Coupon-bearing bonds

A **fixed-coupon bond** is a financial instrument that pays the holder deterministic (known at time $t \leq T_0$) amounts $c_1, \ldots, c_n$, referred to as **coupon payments**, at times $T_1, \ldots, T_n$, where $T_0 < T_1 < \cdots < T_n$. At maturity, time T_n, the holder receives the notional or face value N in addition to the final coupon c_n. Computing the price of a fixed-coupon bond is simply a matter of discounting each cash flow back to time t. The value of a fixed-coupon bond at time $t \leq T_0$, which we denote by $\mathbf{B}_{\text{fixed}}(t)$, is given by

$$\mathbf{B}_{\text{fixed}}(t) = \sum_{i=1}^{n} c_i B(t, T_i) + NB(t, T_n). \tag{1.12}$$

Coupons are typically quoted in terms of a fixed annualised rate of return K, known as the **coupon rate**. Each coupon is then defined as $c_i = \tau_i NK$ for $i = 1, \ldots, n$, where $\tau_i = T_i - T_{i-1}$.

A **floating-coupon bond** or **floating-rate note** is analogous to a fixed-coupon bond with the important difference that the coupon payment at time T_i for $i = 1 \ldots, n$ is a function of the spot LIBOR rate $L(T_{i-1}, T_i)$, which is unknown (stochastic) at time $t < T_{i-1}$. For $i = 1, \ldots, n$ the coupon payment

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