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Joseph Liouville
1809–1882
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# Number Theory in the Spirit of Liouville

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Dedicated to my granddaughter
ISABELLE SOFIE OLSEN
born January 11, 2009

and

to the memory of
JOSEPH LIOUVILLE
March 24, 1809–September 8, 1882

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# Preface

In a series of eighteen papers published between the years 1858 and 1865 the French mathematician Joseph Liouville (1809–1882) introduced a powerful new method into elementary number theory. Liouville's idea was to give a number of elementary (but not simple to prove) identities from which flowed many number-theoretic results by specializing the functions involved in the formulae.

Although Liouville's ideas are now 150 years old, they still do not usually form part of a standard course in elementary number theory. Moreover there is no book in English devoted entirely to Liouville's method, and, although some elementary number theory texts devote a chapter to Liouville's ideas, most do not. In this book we hope to remedy this situation by providing a gentle introduction to Liouville's method. We will not give a comprehensive treatment of all of Liouville's identities but rather give a sufficient number of his identities in order to provide elementary arithmetic proofs of such number-theoretic results as the Girard-Fermat theorem, a recurrence relation for the sum of divisors function, Lagrange's theorem, Legendre's formula for the number of representations of a nonnegative integer as the sum of four triangular numbers, Jacobi's formula for the number of representations of a positive integer as the sum of eight squares, and many others. We will also treat some of the more recent results that have been obtained using Liouville's ideas.

Liouville's method, although beautiful and arithmetic, is still an elementary one and as such has its limitations. As it is based on a number of identities, in order to obtain a particular number-theoretic result using it, the right identity has to be chosen as well as the right choice of the function occurring in it. And this is not always easy to do! Also, as with any elementary method, there are boundaries to what it can achieve. Indeed there are number-theoretic formulae which cannot be proved by Liouville's method and other tools are required to prove them. However, on the other hand, although we do not know the limitations of Liouville's approach, there are still new number-theoretic formulae waiting to be discovered and proved by Liouville's method. Hopefully, after reading this book, the reader will find some.

The prerequisites for this book include the basics of elementary number theory such as divisibility, primes, the fundamental theorem of arithmetic, quadratic reciprocity, the Legendre-Jacobi-Kronecker symbol, and a little about the representation of integers by binary quadratic forms such as $x^2 + xy + y^2$, $x^2 + y^2$ and $x^2 + 2y^2$. Hopefully in reading this book, the reader will enjoy and appreciate the elegant arithmetic proofs that Liouville's method enables us to give. After reading this book the interested reader is encouraged to study the theory of modular forms, where formulae similar to but deeper than the ones given in this book can be found.

The author is grateful to his colleagues A. Alaca and S. Alaca for their comments on the draft of this book, and to M. Huband for her help with Chapter 1. The author is also grateful for the suggestions and corrections that he received from B. C. Berndt of the University of Illinois. He also acknowledges the kindness of Professor Berndt in allowing him to name this book in a similar fashion to Berndt's excellent book "Number Theory in the Spirit of Ramanujan." He also thanks his wife Carole for her help with the references and index.

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Ottawa, Ontario, Canada
March 2010

# Notation

$$\begin{aligned}
\mathbb{N} &= \text{set of positive integers} = \{1, 2, 3, \ldots\} \\
\mathbb{N}_0 &= \text{set of nonnegative integers} = \{0, 1, 2, 3, \ldots\} \\
\mathbb{Z} &= \text{set of all integers} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \\
\mathbb{Q} &= \text{set of all rational numbers} \\
\mathbb{R} &= \text{set of all real numbers} \\
\mathbb{C} &= \text{set of all complex numbers} \\
\mathrm{Re}(z) &= \text{real part of } z \in \mathbb{C}, \text{ that is } \mathrm{Re}(z) = x, \\
&\phantom{=\ } \text{where } z = x + iy,\ x, y \in \mathbb{R} \\
\mathrm{Im}(z) &= \text{imaginary part of } z \in \mathbb{C}, \text{ that is } \mathrm{Im}(z) = y, \text{ where} \\
&\phantom{=\ } z = x + iy,\ x, y \in \mathbb{R} \\
B_n &= n\text{-th Bernoulli number} \\
&\phantom{=\ } \left(B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6}, \ldots.\right) \\
d \mid n &\phantom{=\ } \text{the integer } d \text{ divides the integer } n \\
d \nmid n &\phantom{=\ } \text{the integer } d \text{ does not divide the integer } n \\
p^a \parallel n &\phantom{=\ } \text{the prime } p \text{ is such that } p^a \mid n \text{ and } p^{a+1} \nmid n \\
\gcd(m, n) &= \text{greatest common divisor of the integers } m \text{ and } n \text{ (not both zero),} \\
&\phantom{=\ } \text{which we abbreviate to } (m, n) \text{ if space requires} \\
[x] &= \text{the greatest integer less than or equal to the real number } x \\
\emptyset &= \text{the empty set}
\end{aligned}$$

$$F_k(n) = \begin{cases} 1, & \text{if } k \mid n, \\ 0, & \text{if } k \nmid n. \end{cases}$$

$$G_2(\ell) = \begin{cases} 0, & \text{if } 2 \mid \ell, \\ 1, & \text{if } 2 \nmid \ell. \end{cases}$$

$$s(n) = \begin{cases} 1, & \text{if } n \text{ is a perfect square,} \\ 0, & \text{otherwise.} \end{cases}$$

$$\sigma_k(n) = \begin{cases} \displaystyle\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \notin \mathbb{N}. \end{cases}$$

$$d(n) = \sigma_0(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} 1 = \text{number of positive divisors of } n \in \mathbb{N}$$

$$\sigma(n) = \sigma_1(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d = \text{sum of positive divisors of } n \in \mathbb{N}$$

$$\sigma_k^*(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n \\ n/d \text{ odd}}} d^k$$

$$d^*(n) = \sigma_0^*(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n \\ n/d \text{ odd}}} 1$$

$$\sigma^*(n) = \sigma_1^*(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n \\ n/d \text{ odd}}} d$$

$$d_{k,m}(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n \\ d \equiv k \pmod{m}}} 1$$

$$A(n) = \{(i, j, k) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} \mid i^2 + jk = n,\ k \text{ odd}\}$$

$$A_k(n) = \sum_{\substack{m \in \mathbb{N} \\ 1 \le m < n/k}} \sigma(m)\sigma(n - km)$$

$$r_k(n) = \text{card}\{(x_1, \ldots, x_k) \in \mathbb{Z}^k \mid n = x_1^2 + \cdots + x_k^2\}$$

$$p_k(n) = \text{card}\left\{(x_1, \ldots, x_k) \in \mathbb{Z}^k \mid n = x_1^2 + \cdots + x_k^2, \gcd(x_1, \ldots, x_k) = 1\right\}$$

$$s_{2k}(n) = \text{card}\left\{(x_1, \ldots, x_{2k}) \in \mathbb{Z}^{2k} \mid n = x_1^2 + x_1x_2 + x_2^2 + \cdots + x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2\right\}$$

$$t_k(n) = \text{card}\left\{(x_1, \ldots, x_k) \in \mathbb{N}_0^k \mid n = \tfrac{1}{2}x_1(x_1 + 1) + \cdots + \tfrac{1}{2}x_k(x_k + 1)\right\}$$

$$r(n) = \text{card}\,\{(x, y) \in \mathbb{N}^2 \mid n = x^2 + y^2\}$$

$$R_1(n) = \text{card}\,\{(x, y, z, t) \in \mathbb{N}^4 \mid n = x^2 + y^2 + z^2 + 2t^2\}$$

$$R_2(n) = \text{card}\,\{(x, y, z, t) \in \mathbb{N}^4 \mid n = x^2 + y^2 + 2z^2 + 2t^2\}$$

$$R_3(n) = \text{card}\,\{(x, y, z, t) \in \mathbb{N}^4 \mid n = x^2 + 2y^2 + 2z^2 + 2t^2\}$$

$$\left(\frac{d}{n}\right) = \text{Legendre-Jacobi-Kronecker symbol, which is defined for } d \in \mathbb{Z} \text{ with } d \equiv 0, 1 \pmod 4 \text{ and } n \in \mathbb{N} \text{ (} d \text{ is called the discriminant)}$$

$$\left(\frac{-3}{n}\right) = \begin{cases} +1, & \text{if } n \equiv 1 \pmod 3, \\ -1, & \text{if } n \equiv 2 \pmod 3, \\ 0, & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

$$\left(\frac{-4}{n}\right) = \begin{cases} +1, & \text{if } n \equiv 1 \pmod 4, \\ -1, & \text{if } n \equiv 3 \pmod 4, \\ 0, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

$$\left(\frac{-7}{n}\right) = \begin{cases} +1, & \text{if } n \equiv 1, 2, 3 \pmod 7, \\ -1, & \text{if } n \equiv 3, 5, 6 \pmod 7, \\ 0, & \text{if } n \equiv 0 \pmod 7. \end{cases}$$

$$\left(\frac{-8}{n}\right) = \begin{cases} +1, & \text{if } n \equiv 1 \text{ or } 3 \pmod 8, \\ -1, & \text{if } n \equiv 5 \text{ or } 7 \pmod 8, \\ 0, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

$$\left(\frac{8}{n}\right) = \begin{cases} +1, & \text{if } n \equiv \pm 1 \pmod 8, \\ -1, & \text{if } n \equiv \pm 3 \pmod 8, \\ 0, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

$$s(j,k) = \begin{cases} +1, & \text{if } j \equiv k \equiv 0 \pmod 2, \\ -1, & \text{otherwise.} \end{cases}$$

$$S_{e,f}(n) = \sum_{m=1}^{n-1} \sigma_e(m)\sigma_f(n-m)$$

$$T_{e,f,g}(n) = \sum_{\substack{m \in \mathbb{N} \\ m < n/g}} \sigma_e(m)\sigma_f(n-gm)$$

$$W_{a,b}(n) := \sum_{\substack{m \in \mathbb{N} \\ m < n \\ m \equiv a (\text{mod b})}} \sigma(m)\sigma(n-m)$$

$$S(A,B,C,D,f;n) = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ Cax + Dby = n}} (f(Aa - Bb) - f(Aa + Bb))$$

$$\mu(n) = \text{Möbius function} = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n = p_1 p_2 \ldots p_k, \text{ where } p_1, \ldots, p_k \text{ are distinct primes}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\tilde{\sigma}_s(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} (-1)^{d-1} d^s = \sigma_s(n) - 2^{s+1}\sigma_s(n/2)$$

$$\hat{\sigma}_s(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} (-1)^{n/d-1} d^s = \sigma_s(n) - 2\sigma_s(n/2)$$

$$\tilde{\sigma}(n) := \tilde{\sigma}_1(n) = \sigma(n) - 4\sigma(n/2)$$

$$\hat{\sigma}(n) := \hat{\sigma}_1(n) = \sigma(n) - 2\sigma(n/2)$$

$$d_1(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d = \sigma(n)$$

$$d_2(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n \\ 2 \nmid n}} d = \sigma(n) - 2\sigma(n/2)$$

$$d_3(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n \\ 2 \mid n}} d = 2\sigma(n/2)$$

$$d_4(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n \\ 2 \nmid n/d}} d = \sigma(n) - \sigma(n/2)$$

$$d_5(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n \\ 2 \mid n/d}} d = \sigma(n/2)$$

$$d_6(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} (-1)^{d-1} d = \sigma(n) - 4\sigma(n/2)$$

$$d_7(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} (-1)^{n/d-1} d = \sigma(n) - 2\sigma(n/2)$$

$$D(r, s; n) := \sum_{m=1}^{n-1} d_r(m) d_s(n-m)$$

$$\begin{aligned}\Delta &:= \text{set of triangular numbers}\\ &= \{0, 1, 3, 6, 10, 15, \ldots\}\end{aligned}$$

$$R(n) := \operatorname{card}\{(t_1, t_2, t_3, t_4) \in \Delta^4 \mid n = t_1 + t_2 + 2t_3 + 2t_4\}$$

$$\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$$

$$SL_2(\mathbb{Z}) := \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1\right\}$$

$$E_k(q) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \quad q = e^{2\pi i z},\ z \in \mathbb{H},\ k(\text{even}) \geq 2$$

$$M_k(SL_2(\mathbb{Z})) := \text{space of modular forms of weight } k \text{ for } SL_2(\mathbb{Z})$$