REAL ANALYSIS THROUGH MODERN INFINITESIMALS

Real Analysis through Modern Infinitesimals provides a course in mathematical analysis based on internal set theory (IST), introduced by Edward Nelson in 1977. After motivating IST through an ultrapower construction, the book provides a careful development of this theory, representing each external class as a proper class. This foundational discussion, which is presented in the first two chapters, includes an account of the basic internal and external properties of the real number system as an entity within IST. In its remaining 14 chapters, the book explores the perspective offered by IST as a wide range of real analysis topics are surveyed. The topics thus developed begin with those usually discussed in an advanced undergraduate analysis course and gradually move to those that are suitable for more advanced readers.

This book may be used for reference, self-study, and as a source for advanced undergraduate or graduate courses.

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Real Analysis through Modern Infinitesimals

NADER VAKIL
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To my mother and father
and
to Farideh and Parmede

CONTENTS

Cambridge University Press
978-1-107-00202-9 - Real Analysis through Modern Infinitesimals
Nader Vakil
Frontmatter
More information

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PREFACE

The discrete and the continuous are among the most fundamental categories of the human mind, and our urge to create theories that connect the two has prompted us to invent and deploy the *infinite set* in a monumental intellectual endeavor known as *mathematical analysis.*

As a branch of mathematics, analysis has evolved during the last four centuries. Prior to this time, mathematics was mainly geometry and arithmetic (together with some algebra). Natural numbers were the primary concepts of arithmetic, which provided for a quantitative study of discrete phenomena; and straight lines, curves, surfaces, etc. were the primary concepts of geometry, which provided for a quantitative study of continuous phenomena. So from a historical point of view, an understanding of the continuous in terms of the discrete could mean none other than constructing analytic models of the primary concepts of geometry using the stuff of arithmetic, which we accomplished under the auspices of our infinite sets.

The familiar real number system is *one* example of an analytic model of the geometric line, and the familiar system of the complex numbers is *one* example of an analytic model of the plane. Although these number systems are now (as far as we know) free from contradictions, it is a well-known piece of history that the construction of a real number system which is not fraught with contradictions and which represents the geometric line as an assemblage of infinitely many geometric points while capturing our intuition of its continuity was a formidable task whose accomplishment was the result of the accumulated contributions of the most creative minds of our species.

So why was a fitting and flawless analytic model of the line such an elusive construct? Remember that the concept of a non-denumerable infinite set is characteristic of such models, and infinity was an elusive concept. Indeed, there is little that is concrete or intrinsically natural about infinite sets yet they are indispensable to constructing analytic models of continuous quantities. This means that much of our intuition about what appears continuous to us must be represented as set-theoretical constructs that use infinite sets as building blocks. Many of these constructs and those obtained from them by higher levels of abstraction are quite intricate, and the theories we develop as we explore their properties and interrelationships require the employment of logic at levels of syntactical complexity that rarely occur in other areas of human discourse.

This state of affairs can be ameliorated by upgrading the set-theoretical framework of mathematical analysis so as to allow for a treatment of the subject that employs the *method of modern infinitesimals*, or *nonstandard analysis*. This upgrading will supplement mathematical analysis with methods that are remarkably simple in both conceptual and logical terms.[1]

The mathematical ideas needed for constructing such upgraded frameworks are manufactured in an important branch of mathematical logic called model theory. Model theory is essential for understanding how the basic properties of these frameworks derive from the manner in which they are constructed. Indeed, readers who are interested in the foundation of modern infinitesimals should study model theory.

But this book has a different aim. Our goal here is to explore the applications of modern infinitesimals in studying the central topics of real analysis. The framework we need can be described axiomatically without recourse to model theory. The most widely used axiomatic approach to modern infinitesimals is called *internal set theory* (IST),[2] and we shall avail ourselves of these nonstandard methods by choosing IST for the foundation of our development. This theory provides a simple framework for developing modern infinitesimals and can be easily mastered by readers with no background in mathematical logic.

The book's introduction provides an overview of how ultrapower constructions (see Definition 0.5.2) yield structures whose members can be classified as standard or nonstandard and as internal or external sets. This is used to motivate IST as an axiomatic description of the class of internal sets in such structures.

The first chapter gives a careful development of IST in which the groundwork is laid for representing each external class as a proper class. It is worth mentioning that an axiomatic description of ZFC is subsumed in what we present as the foundation of our work, for IST is an extension of Zermelo–Fraenkel set theory (ZFC). The axioms of IST consist of the axioms of ZFC plus three additional axioms. The first of these – the transfer axiom – is described in Chapter 1. The other two – the idealization and standardization axioms – together with their basic

[1] Readers who are interested in the philosophical issues concerning continuity and discreteness are invited to read an interesting book, *The Continuous and the Infinitesimal in Mathematics and Philosophy*, by John L. Bell, Polimetrica 2005, which offers a thorough account of the historical development of these concepts.

[2] IST was introduced by Edward Nelson (Princeton University) in an article published in 1977 in the *Bulletin of the American Mathematical Society*, **83**, 1165–1198. In the same article, Nelson proved that a model of IST can be constructed within ZFC. This implies that IST is a consistent theory provided that ZFC is a consistent theory.

consequences are taken up in Chapter 2. Also in Chapter 2 we develop the basic "internal" and "external" properties of the real number system as an entity within IST.[3]

In the remaining 14 chapters, we explore the perspective offered by IST as we survey a wide range of topics in analysis. The topics we discuss begin with those usually studied in an advanced undergraduate real analysis course and gradually move to topics that are suitable for more advanced readers. This should appeal to readers who want to learn how to apply modern infinitesimals by exploring their role in a great variety of situations.

Going into more detail, the first two chapters are followed by a discussion of real sequences and series, the topology of the real line, and the analysis of real-valued functions of a single real variable, including their continuity, differentiability, and Riemann integrability and the properties of sequences and series of such functions (with an example of a nowhere-differentiable everywhere-continuous function). We also give a more extensive discussion of infinite series. This constitutes Part I of the book.

Part II provides an introduction to point-set topology (including metric spaces and function space topologies), normed vector spaces, and multidimensional (finite as well as infinite) differential calculus. In Chapter 10 we discuss the notions of a basis and subbasis for a topology and use them to introduce the weak topology. Then we specialize the weak topology to get the product topology, which in turn is specialized to yield the Euclidean topology on $\mathbb{R}^n$. This allows us to obtain some topological properties of $\mathbb{R}^n$ as corollaries to theorems concerning the product topology. Other topics in this chapter include convergence, continuity, compactness (an elegant proof of the Tychonov theorem is included), local compactness, and connectedness, all in the context of a general topological space. We end Chapter 10 with a brief discussion of monads of filters and prove two important theorems due to Luxemburg.[4] In Chapter 11 these topological concepts are explored further in the special cases of metric spaces and normed vector spaces. The examples discussed include the space of vector-valued bounded functions under the uniform norm, ℓ_p-spaces with $p \in [1, \infty]$, and $\mathbb{R}^n$ under the p-norm. The results proved include the Cauchy–Schwarz inequality, the Riesz lemma, the Urysohn lemma, and a discussion is given of the Heine–Borel theorem (including the fact that ℓ_2 has a non-compact closed and bounded subset). We close this chapter with a discussion of the nonstandard concept of the "standard hull" of a class and its application to the Peano existence theorem. In Chapter 12 we discuss the following: complete metric spaces; totally bounded spaces; the equivalence of sequential, limit-point,

[3] The analytic model of the line – the real number system – that one develops within the framework of IST has two types of properties: *internal* and *external*. Its internal properties are *formally* identical with the properties of the ordinary real number system, but its external properties are entirely new concepts.

[4] W. A. J. Luxemburg is Emeritus Professor of Mathematics at the California Institute of Technology.

and covering compactness in a metric space; the consequences of the Lipschitz and uniform equivalence of metrics; the completion of a metric space; the metrization of infinite products; Banach spaces; the Hölder inequality and Minkowski inequality for Riemann integrable functions; the completion of normed vector spaces; the Lebesgue space $L_p[a, b]$ and the Sobolev space $W^{m,p}[a, b]$ as the completions of $(C[a, b], +, \cdot, \|\cdot\|_p)$ and $(C^m[a, b], +, \cdot, \|\cdot\|_{W^{m,p}})$, respectively; and the Daniell integral on $[a, b]$. In Chapter 13 we explore some applications of completeness. The results discussed include: the Baire category theorem and the Baire–Osgood theorem with applications; the Tietze extension theorem and the continuous extension of a function that is uniformly continuous on a dense set; and the Banach fixed point theorem. In Chapter 14 we see that all norms on a finite-dimensional vector space are equivalent and turn the space into a Banach space and that these spaces are characterized by the fact that their closed unit balls are compact. Chapter 14 also includes a brief discussion of invertible bounded linear operators with applications to existence and uniqueness theorems for the Fredholm and Volterra integral equations. In Chapter 15 we discuss differential calculus on normed vector spaces, with an emphasis on $\mathbb{R}^n$ and the finite-dimensional case. This chapter includes a very detailed nonstandard proof of the inverse function theorem. Finally, in Chapter 16, we consider function space topologies (including compact-convergence topology and its metrization and compact-open topology and also the connections between these two topologies), the Stone–Weierstrass theorem with applications, and some useful versions of the Ascoli theorem, deriving them from a fairly general variant of this theorem. There are also four appendices, which are included to enhance and affirm the book's character as a self-contained resource.

The text contains more than a thousand exercises. Some are the usual end-of-section or end-of-chapter exercises and some are placed in the body of the text right after a definition or a theorem, to encourage the reader to stop and think about what s/he has just read. These exercises are almost never hard, and hints are provided where a proof may not be quite straightforward. Most of these exercises should be regarded as an integral part of the theory being developed, and as such they should be at least attempted before the reader moves on further in the discussion. Indeed, some exercises will be referred to later in the text.

The text should be accessible to anyone with a background in undergraduate mathematics. In fact, the first six chapters of the book have been used many times here at Western Illinois University (WIU) as a course taken by a mixture of advanced undergraduate students and beginning graduate students. The background of the undergraduate students included a course in elementary set theory and logic intended to prepare students for upper division theoretical courses such as abstract algebra, real analysis, geometry, and topology. The beginning chapters of the text have also been used as a special topics graduate-level course at St Cloud State University in Minnesota. I have also used the material as a one-semester course at graduate level. In general, to provide a one-semester course, with a strong undergraduate real analysis course as a prerequisite, one can use the book as follows.

- As a foundation use Chapters 1 and 2 (leaving on one side Sections 2.1-2.10 and focussing on Sections 2.11–2.16) and Sections 3.1, 4.1, 4.3, 5.1, and 5.5. Some of this material should count as review and some as new. This can all be covered in about three weeks.

Then, on the basis of this foundation, one can construct a number of different one-semester courses to be offered as an honors course, an upper-division seminar, a graduate course, directed reading, or self-study. Here are two examples.

- Sections 10.1–10.6, Chapter 11, Sections 12.1–12.4, and Chapters 14–15.
- Chapters 10–13 and Chapter 16. This selection is suitable for a one-semester course that emphasizes the application of the methods of infinitesimals to point-set topology including function spaces.

Some remarks about the way the topics are developed may be helpful. As in any other analysis course, our principal subject of discussion is the internal theory of analysis (i.e., analysis in its traditional form as developed by Riemann, Weierstrass, Lebesgue, Banach, and others). Infinitesimals are served as a side dish although in fact they are an integral part of the course. Consider, for example, the notion of continuity. In this book, as in many others, students are first introduced to this notion through its $\epsilon\delta$ formulation. Then, in the course of the development of the notion, one proves several alternative equivalent formulations for it (e.g., one involving sequences and another involving open sets). Theorems about continuity can then be proved using any formulation that seems advantageous for the situation at hand. A difference between this book and other texts is that this set of equivalent formulations of a given concept also includes one that involves infinitesimals – our set-theoretical framework allows us to introduce the methods of modern infinitesimals on the side and use them whenever they are advantageous. Here is another example: the concept of the Riemann integral is introduced in this book through its $\epsilon\delta$ formulation, involving Riemann sums. Then we give an equivalent formulation involving infinitesimals and use it to prove two other traditional equivalent formulations, one involving Darboux sums and another involving measure-zero sets. We use the latter to derive most of the nontrivial properties of the Riemann integral.

Similarly, for the mathematical structures we discuss we generally use a traditional method for their construction or description. For example, in Chapter 2 the system $(\mathbb{R}, +, \cdot, <)$ is introduced through its "internal" properties as a complete ordered field. Hence the reader may rightly regard this system as having the same properties as the familiar real number system constructed within the ordinary set theory (ZFC) using, for example, Dedekind cuts or Cauchy sequences of rational numbers. But we also remind our readers that the properties of this system depend not only on its properties as a complete ordered field but also on the properties of the set-theoretical structure of which it is assumed to be a part. Since IST is an extension of ZFC, it has features that ZFC lacks. So we can use the extra features of IST to see that the system $(\mathbb{R}, +, \cdot, <)$ possesses properties that cannot even be formulated when the system is assumed to be an inhabitant of ZFC. The same is true of all the other classical structures, such as $\mathbb{R}^n$, ℓ_∞, $C[a, b]$, $L_p[a, b]$, and $W^{m,p}[a, b]$, that we discuss in this book. Each of these structures is defined, in this text, through its internal properties (i.e., through one of the ways in which they are usually defined). But, because we regard these structures as entities within IST, we can associate with them concepts that cannot be formulated in classical terms.

These nonstandard concepts yield new methods which are not only effective in bringing elegance and economy to the development of mathematical analysis but also produce interesting new results in various branches of both pure and applied mathematics. Indeed, the literature on these methods abounds with success stories in such areas as functional analysis, stochastic analysis, mathematical economics, and mathematical physics.[5] These applications are at once elegant and sophisticated, for they are based on concepts that are simple as well as powerful.

Modern infinitesimal analysis has, over the past 50 years, evolved into a well-established approach to the mathematics of the infinite,[6] but this new paradigm of mathematics is yet to command the broad attention it merits. The author hopes that this volume will motivate and facilitate a more widespread adoption of the methods of modern infinitesimals both in teaching and in mathematical research.[7]

Acknowledgements It is very difficult if not impossible to recount and acknowledge the influence of all the people who have made this book possible. I have to select a few, and I begin by remembering my teacher the late Edwin Hewitt, who influenced me most as my perspective on mathematical analysis and its foundation was taking shape. I would particularly like to thank W. A. J. Luxemburg, who has had the most influence on the way I view nonstandard analysis. I have had the privilege of his guidance and support since the summer of 1983, when I went to Caltech to consult with him regarding my doctoral dissertation on nonstandard analysis. I am, in particular, deeply grateful to him for his kind words of encouragement throughout my work on this project. The influence of Edward Nelson on the foundational aspect of this book is obvious, and his encouragement and support for this project has been warm and generous. I thank my friend and

[5] See *Nonstandard Analysis for the Working Mathematician* (and the references therein) by Peter A. Loeb and Manfred Wolff (eds.), Kluwer Academic, 2000.

[6] Modern infinitesimal analysis (also known as nonstandard analysis) was created by Abraham Robinson (1918–1974) in the early 1960s. For a complete history of nonstandard analysis, we refer the reader to J. W. Dauben's excellent biography of Robinson published by Princeton Press in 1994.

[7] In addition to Robinson's nonstandard analysis, there are other theories of infinitesimals employable in analysis. For example, synthetic differential geometry or smooth infinitesimal analysis, introduced by F. W. Lawvere in the 1960s, provides a different approach to analysis using nilpotent infinitesimals (see, for example, John L. Bell, *A Primer of Infinitesimal Analysis*, second edition, Cambridge University Press, 2008).

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Frontmatter
More information

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I plan to create a website for the book in the future. Please send me errors and suggestions for improvements at the following address:

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