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## Basic Physical Principles and Equations

### 1.1 The Large-Scale Ocean Circulation

Systematic observations of the fluid properties of Earth's ocean, made primarily over the course of the last hundred years, reveal coherent features with scales comparable to those of the ocean basins themselves. These include such structures as the subtropical main thermoclines and anticyclonic gyres, which appear in all five midlatitude ocean basins, and the meridional overturning circulations that support the exchange of waters across the full meridional extent of the ocean, from the polar or subpolar latitudes of one hemisphere to the opposing high latitudes of the other. These features and motions, which prove to be connected by robust dynamical balances, constitute the large-scale circulation of the Earth's ocean.

The global field of long-term mean sea-surface temperature is dominated by the meridional gradients between the warm equatorial regions and the cold poles but contains significant zonal gradients as well (Figure 1.1). The global long-term mean sea-surface salinity field has a more complex structure (Figure 1.2), with isolated maxima in the evaporative centers of the midlatitude subtropical gyres. The global sea-surface density field computed from long-term mean temperature and salinity reflects the competing influences of temperature and salinity on density (Figure 1.3). These fields are the surface expressions of complex three-dimensional interior property fields. The downward penetration of the warm equatorial temperatures is generally limited to the upper one-fifth of the water column (Figure 1.4), while salinity perturbations are less strongly confined to shallow depths (Figure 1.5). Lateral density gradients, more important for the large-scale circulation than vertical density gradients, largely reflect temperature variations in the upper subtropics and salinity variations at high latitudes and great depths (Figure 1.6). Coherent, large-scale structures in these property fields include the mean downward slope toward the west of isothermal and isohaline surfaces in the subtropics and the associated eastward zonal gradients of density (Figure 1.7). These distributions support and are in part maintained by coherent,

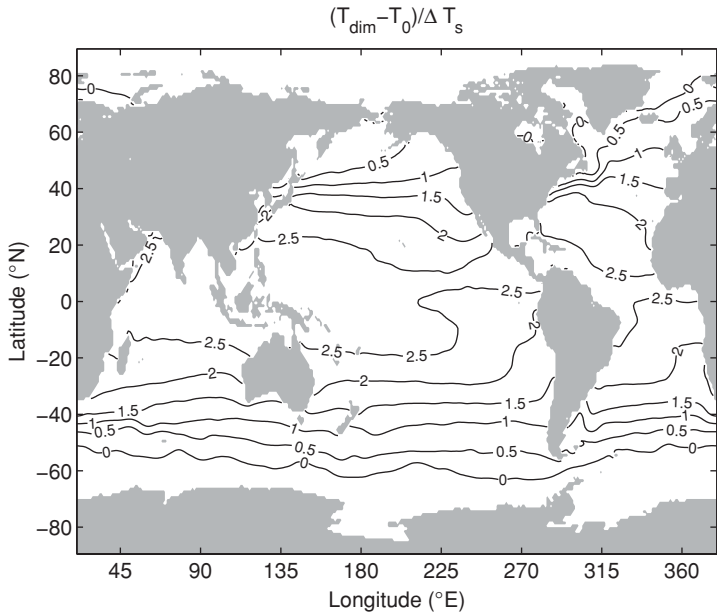


Figure 1.1. Dimensionless long-term mean sea-surface temperature  $T = (T_{\text{dim}} - T_0) / \Delta T_s$  vs. longitude ( $^{\circ}\text{E}$ ) and latitude ( $^{\circ}\text{N}$ ) for the world ocean, where  $T_{\text{dim}}$  is the observed temperature (K),  $T_0 = 275$  K, and  $\Delta T_s = 10$  K.

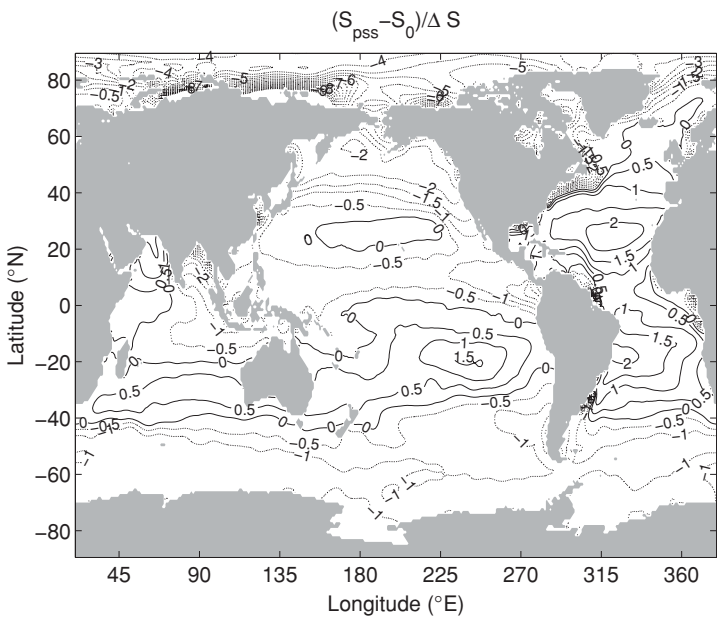


Figure 1.2. Dimensionless long-term mean sea-surface salinity  $S = (S_{\text{pss}} - S_0) / \Delta S$  vs. longitude ( $^{\circ}\text{E}$ ) and latitude ( $^{\circ}\text{N}$ ) for the world ocean, where  $S_{\text{pss}}$  is the observed salinity,  $S_0 = 35$ , and  $\Delta S = 1$  for salinity on the practical salinity scale.

1.1 The Large-Scale Ocean Circulation

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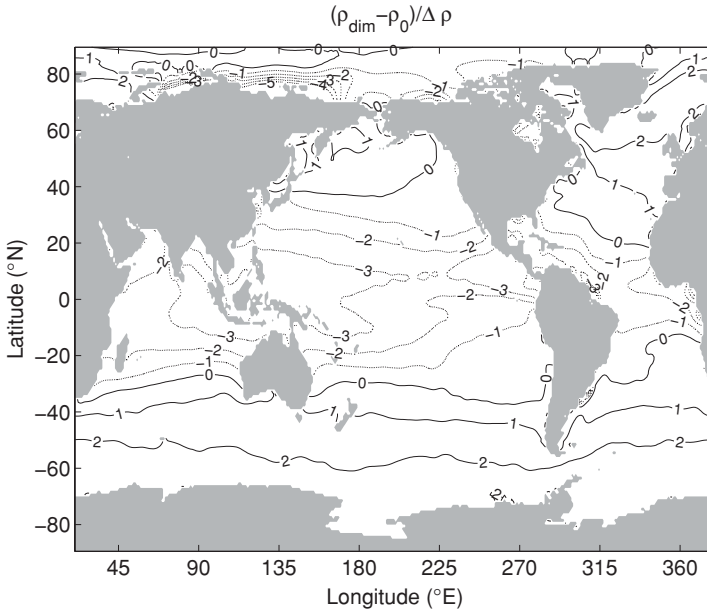


Figure 1.3. Dimensionless long-term mean sea-surface density  $\rho = (\rho_{\text{dim}} - \rho_0)/\Delta\rho$  vs. longitude ( $^{\circ}\text{E}$ ) and latitude ( $^{\circ}\text{N}$ ) for the world ocean, where  $\rho_{\text{dim}}$  is the observed density ( $\text{kg m}^{-3}$ ),  $\rho_0 = 1025 \text{ kg m}^{-3}$ , and  $\Delta\rho = 1 \text{ kg m}^{-3}$ .

large-scale fluid motions, which in turn form one component of the spectrum of motions that is observed to occur throughout the world ocean.

The broad range of spatial and temporal scales over which ocean fluid motion occurs, together with the intrinsic nonlinearities of the general fluid dynamical equations, makes the direct analytical or numerical solution of complete fluid models of ocean circulation utterly intractable. However, theoretical progress can be made by using observed knowledge of the general characteristics and scales of observed features and motions to guide appropriate simplification of these general equations. For the large-scale circulation, a basic set of equations—the planetary geostrophic equations—can be deduced from a relatively limited set of a priori assumptions regarding the scales of the motion. From a theoretical point of view, this approach is essentially deductive in nature: it results in a system of equations for the purely large-scale motion that does not depend on assumptions about, or models of, the mean effect of smaller-scale motions on the large scale. In their simplest form, these planetary geostrophic equations are equivalent to a single partial-differential evolution equation that is first order in time but higher order in space.

Unfortunately, for the theoretician, this deductive model is not closed: it is not possible to pose a well-defined general boundary or initial-boundary value problem for these equations without introducing additional assumptions regarding small-scale motion near boundaries. The theory of large-scale ocean circulation thus rests on a

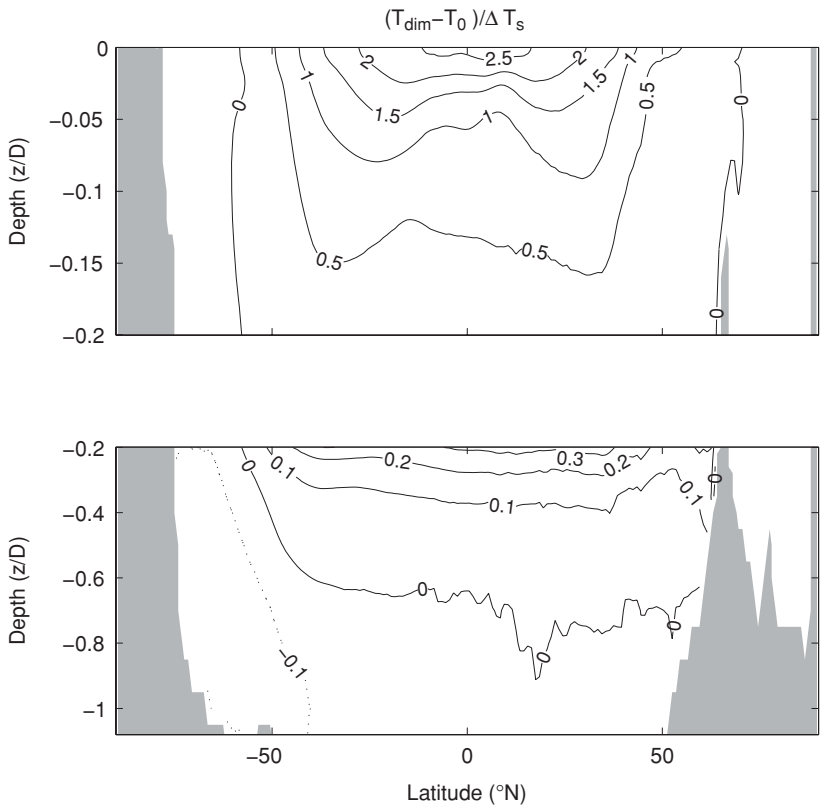


Figure 1.4. Dimensionless zonally averaged cross sections of long-term mean in situ temperature  $T$  vs. latitude ( $^{\circ}\text{N}$ ) and depth  $z/D$  for the world ocean, for dimensionless  $T$  defined as in Figure 1.1 and  $D = 5000\text{ m}$ : (top)  $-0.2 < z/D < 0$ ; (bottom)  $z/D < -0.2$ . The vertical coordinate  $z$  is defined as positive upward, with  $z = 0$  at the sea surface so that the ocean is confined to  $z < 0$ .

combination of two fundamentally different elements: an essentially deductive set of large-scale equations and a supplemental set of models or assumptions describing the mean effect of smaller-scale motion on the large scale. The approach taken here is to proceed deductively as far as possible and then to introduce the minimal set of supplemental models or assumptions required to obtain theories that are sufficiently complete to offer the essential insights into ocean circulation that are sought.

1.2 Physical Variables

In the continuum representation of the substance and flow of seawater that is the appropriate starting point for theories of large-scale ocean circulation, there are seven basic physical fields of interest, each of which is a function of time  $t$  and of position  $\mathbf{x}$  in the three-dimensional space in which the ocean resides. These physical fields are mass density  $\rho$ , three-component vector velocity  $\mathbf{v}$ , pressure  $p$ , thermodynamic

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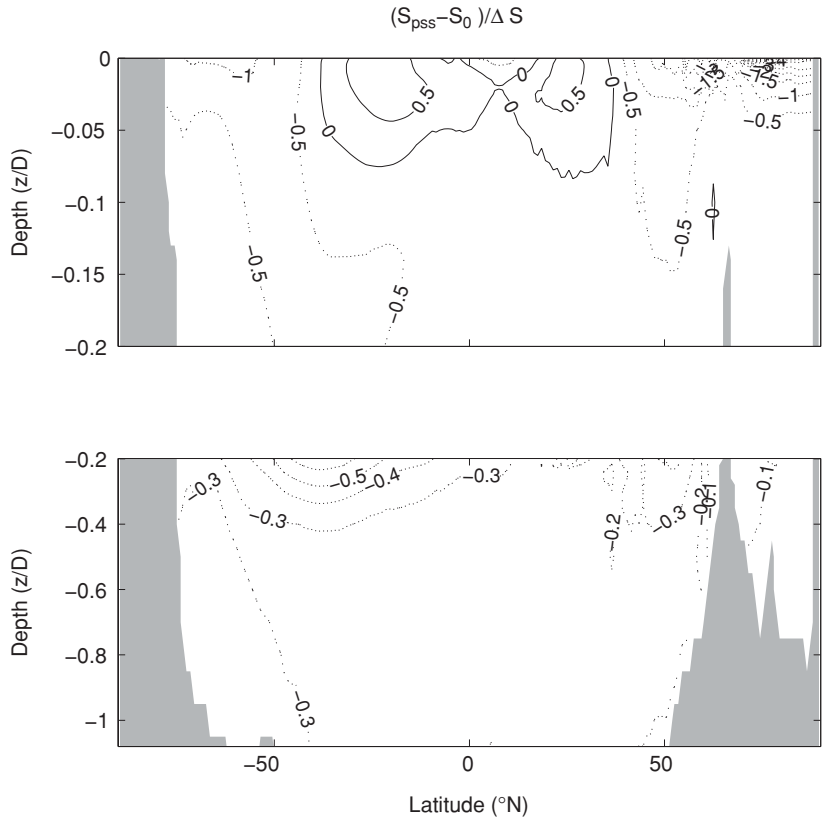


Figure 1.5. Dimensionless zonally averaged cross sections of long-term mean salinity  $S$  vs. latitude ( $^{\circ}\text{N}$ ) and depth  $z/D$  for the world ocean, for dimensionless  $S$ , defined as in Figure 1.2, and  $z/D$ , defined as in Figure 1.4: (top)  $-0.2 < z/D < 0$ ; (bottom)  $z/D < -0.2$ .

energy  $\hat{e}$ , and salinity  $S$ . From a mathematical point of view, these physical fields form a complete set of dependent variables that describes the ocean state.

The fluid velocity  $\mathbf{v}$  is itself best defined physically as a mass-weighted mean velocity of an infinitesimal fluid parcel. Two different pressures may be defined: the dynamic pressure and the thermodynamic pressure; following a standard approximation, it is assumed here that the fluid is locally in thermodynamic equilibrium and that these two pressures are equivalent. The thermodynamic energy  $\hat{e}$  is a function of the pressure  $p$ , temperature  $T$ , and salinity  $S$ , or an equivalent set of three thermodynamic variables; this function is the equation of state for seawater and encodes the physical properties of the fluid substance. The salinity  $S$  represents the concentration of a mixture of chemical salts whose relative proportions are set largely by chemical weathering of exposed continental forms and may be taken as uniform throughout the world ocean. In some cases, it can prove useful to exchange dependent and independent variables; for example, it can be convenient to use pressure as a pseudo-vertical coordinate.

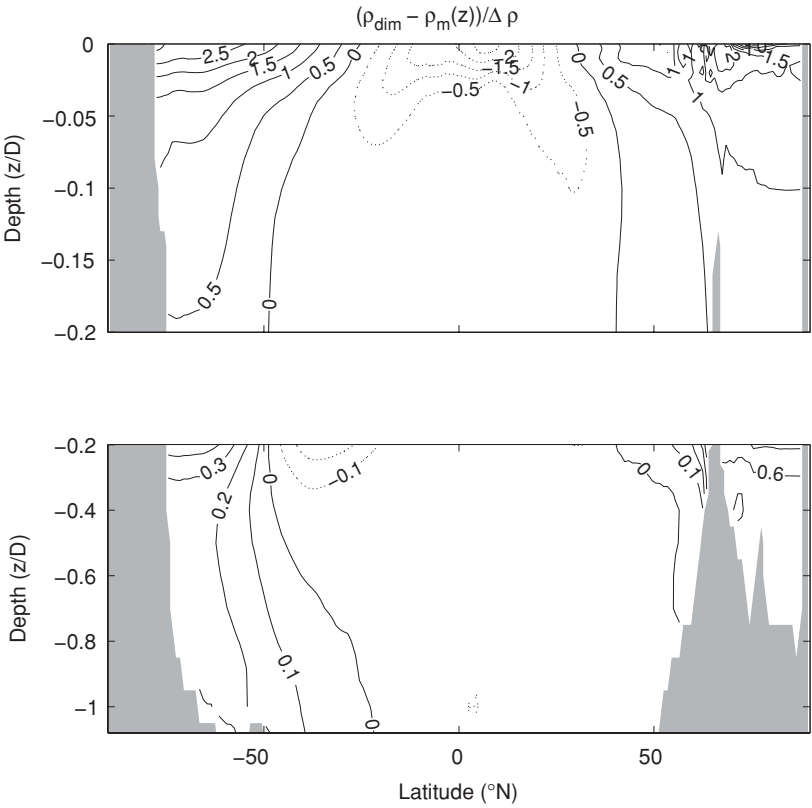


Figure 1.6. Dimensionless zonally averaged cross sections of long-term mean in situ density  $\rho = [\rho_{\text{dim}} - \rho_m(z)]/\Delta\rho$ , defined as the departure from its horizontally averaged mean at each depth  $[\rho_m(z)]$ , scaled by  $\Delta\rho = 1 \text{ kg m}^{-3}$ , vs. latitude ( $^{\circ}\text{N}$ ) and depth  $z/D$  for the world ocean, for  $z/D$  as in Figure 1.4: (top)  $-0.2 < z/D < 0$ ; (bottom)  $z/D < -0.2$ .

The equations that form the basis for the circulation theories are derived from conservation, or balance, principles for mass, momentum, salt, and thermodynamic energy, plus the empirically determined equation of state for seawater. These classical physical principles and the corresponding equations are briefly reviewed in the following sections. The corresponding derivations can be found in standard fluid mechanics texts.

1.3 Fluid Motion and the Material Derivative

The motion of a fluid in two- or three-dimensional space may be described by a two- or three-dimensional vector function or map  $\mathbf{X}(\mathbf{a}, \tau)$  that gives, for each  $\mathbf{a}$  in the domain, the position  $\mathbf{x} = \mathbf{X}(\mathbf{a}, \tau)$  at time  $\tau$  of the fluid parcel with initial position  $\mathbf{x} = \mathbf{a}$  at  $\tau = 0$ . This is the so-called *Lagrangian* description, in which labeled fluid

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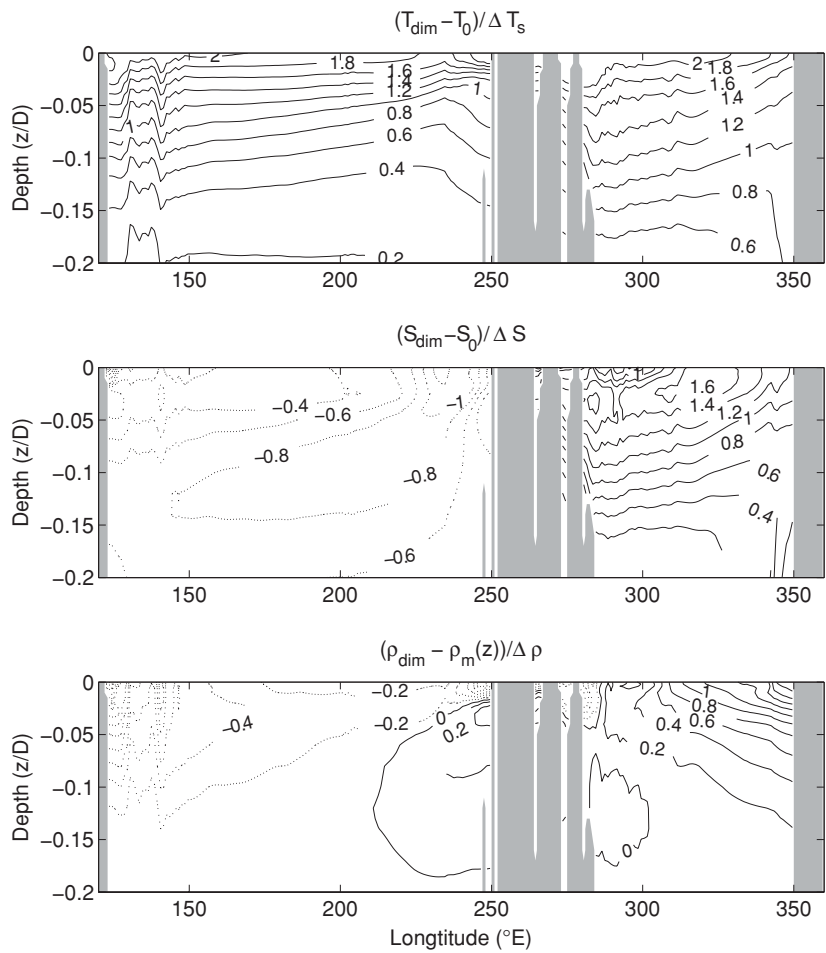


Figure 1.7. Dimensionless meridionally averaged cross sections of long-term mean in situ temperature, salinity, and density  $\rho = [\rho_{\text{dim}} - \rho_m(z)] / \Delta\rho$  vs. longitude ( $^{\circ}\text{E}$ ) and depth  $z/D$  for the northern hemisphere subtropics ( $15^{\circ}\text{N}$ – $45^{\circ}\text{N}$ ) and  $-0.2 < z/D < 0$ . The dimensionless  $T$ ,  $S$ , and  $\rho$  are defined as in Figures 1.1, 1.2, and 1.6, respectively, and  $z/D$  is defined as in Figure 1.4.

parcels are followed as they move. The Lagrangian fluid velocity  $\mathbf{v}$  is then just the partial derivative of the map  $\mathbf{X}$  with respect to time  $\tau$ , holding the initial positions  $\mathbf{a}$  fixed:

$$\mathbf{v}(\mathbf{a}, \tau) = \frac{\partial}{\partial \tau} \mathbf{X}(\mathbf{a}, \tau). \tag{1.1}$$

The inverse  $\mathbf{A}(\mathbf{x}, t)$  of  $\mathbf{X}$  gives the initial position  $\mathbf{a}$  of the fluid parcel with position  $\mathbf{x}$  at time  $\tau = t$ :

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}[\mathbf{X}(\mathbf{a}, \tau = t), t] = \mathbf{a}. \tag{1.2}$$

Alternatively, the fluid motion may be described by specifying the fluid velocity directly, at each point in space and time. This is the *Eulerian* description, in which the motion is considered relative to a fixed reference frame, and no parcel labels are required. The corresponding Eulerian fluid velocity field  $\mathbf{u}(\mathbf{x}, t)$  is simply related to (1.1) and (1.2):

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{v}[\mathbf{A}(\mathbf{x}, t), t] = \frac{\partial \mathbf{X}}{\partial \tau}[\mathbf{A}(\mathbf{x}, t), t]. \quad (1.3)$$

Note that the partial derivative  $\partial \mathbf{X} / \partial \tau$  in (1.3) is still taken with respect to  $\tau$  while holding  $\mathbf{a}$  fixed, as in (1.1), and not with respect to  $t$  while holding  $\mathbf{x}$  fixed. Both  $t$  and  $\tau$  represent the same time variable, and their values are the same; the reason for the different symbols is this distinction between the meaning of the partial derivatives.

It turns out that the dynamical equations for fluid motion can often be posed in the Eulerian framework, with no, or limited, specification of the Lagrangian pathways of fluid parcels. In that case, the fluid velocity  $\mathbf{u}(\mathbf{x}, t)$  is most conveniently defined as the ratio of the momentum to mass density of an infinitesimal fluid parcel or, equivalently, as a mass-weighted mean velocity of the fluid parcel. A Lagrangian representation (1.1) consistent with this definition can be obtained by modifying the initial Lagrangian label field  $\mathbf{a}$  so that mass is uniformly distributed in  $\mathbf{a}$  space.

Consider a scalar fluid property  $\hat{F}(\mathbf{a}, \tau)$ , which might be the concentration of a dissolved substance such as salinity. The rate of change of the value of this property for a given fluid parcel is the partial derivative of  $\hat{F}$  with respect to  $\tau$  while holding  $\mathbf{a}$  fixed, and is denoted  $D\hat{F}/Dt$ :

$$\frac{D\hat{F}}{Dt} = \frac{\partial}{\partial \tau} \hat{F}(\mathbf{a}, \tau). \quad (1.4)$$

Suppose that  $\hat{F}$  is given, instead, in terms of its distribution  $F(\mathbf{x}, t) = \hat{F}[\mathbf{A}(\mathbf{x}, t), t]$  with respect to  $\mathbf{x}$  at time  $t$ . Then the derivative following the motion, as a function of  $\mathbf{x}$  and  $t$ , can be computed using the chain rule:

$$\frac{D\hat{F}}{Dt} = \frac{D}{Dt} F(\mathbf{x}, t) = \left( \frac{\partial}{\partial \tau} F[\mathbf{X}(\mathbf{a}, \tau), \tau] \right)_{\tau=t} = \mathbf{u} \cdot \nabla F + \frac{\partial F}{\partial t}. \quad (1.5)$$

Here  $\partial F / \partial t$  denotes the local rate of change of  $F$  with spatial coordinates  $\mathbf{x}$  held fixed, and  $\nabla$  is the three-dimensional gradient operator in  $\mathbf{x}$  space so that  $\nabla F = \text{grad } F$  is the  $\mathbf{x}$  gradient of  $F$ . The operator  $D/Dt$  defined by (1.4) or (1.5) is the material derivative: the rate of change of a fluid property following the motion of the fluid.

## 1.4 Mass Conservation

The principle of conservation of mass demands that fluid mass be neither created nor destroyed and relates the mass density to the motion. With the continuum representation of the fluid, this principle can be expressed as a partial differential equation that



## 1.5 Momentum Balance

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balances the rate of change of the mass density  $\rho$  and the divergence of the mass flux  $\rho \mathbf{u}$ :

$$\partial \rho / \partial t + \nabla \cdot \rho \mathbf{u} = \partial \rho / \partial t + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \quad (1.6)$$

Here  $\nabla \cdot \mathbf{F} = \text{div } \mathbf{F}$  is the divergence of a vector field  $\mathbf{F}$ . Equation (1.6) is the Eulerian expression of the principle of mass conservation.

In the Lagrangian setting, the principle of mass conservation is

$$\rho J_a = \rho_0. \quad (1.7)$$

Here  $\rho_0$  is the initial mass density, assumed constant in  $\mathbf{a}$  space, and  $J_a$  is the Jacobian determinant:

$$J_a = \det \left( \frac{\partial \mathbf{X}}{\partial \mathbf{a}} \right) = \det \left( \left\{ \frac{\partial X_i}{\partial a_j} \right\} \right), \quad (1.8)$$

where  $\partial X_i / \partial a_j$  is the matrix of partial derivatives of fluid-parcel positions  $\mathbf{X}(\mathbf{a}, \tau)$  with respect to their initial positions  $\mathbf{a}$ . Equation (1.7) follows from the me transformation rule of integral calculus.

## 1.5 Momentum Balance

Newton's third law of motion, a basic principle of classical physics, relates the rate of change of momentum of a particle to the sum of applied forces. In a reference frame rotating with angular velocity  $\boldsymbol{\Omega}$ , the corresponding statement for a fluid may be written

$$\rho \left( \frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} \right) = -\nabla p - \rho \nabla \Phi_g - \nabla \cdot \mathbf{d}, \quad (1.9)$$

where  $\Phi_g$  is a potential for the sum of the gravitational and centrifugal forces and  $\nabla \cdot \mathbf{d}$  represents the component-wise divergence  $\partial d_{ij} / \partial x_j$  of the deviatoric stress tensor  $\mathbf{d}$ . In Cartesian coordinates  $\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$ ,

$$(\mathbf{d})_{ij} = d_{ij} = -2\mu \left( e_{ij} - \frac{1}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right), \quad (1.10)$$

where

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (1.11)$$

In (1.10),  $\mu$  is the dynamic viscosity, and  $\delta_{ij}$  is the Kronecker  $\delta$  function.

The term in (1.9) proportional to  $\boldsymbol{\Omega}$  is the *Coriolis force*. For the rotating Earth,  $|\boldsymbol{\Omega}| = \Omega = 7.29 \times 10^{-5} \text{ s}^{-1}$ , and it will be sufficient here to take  $\Phi_g = gz$ , where  $z$  is the local vertical coordinate and  $g = 9.81 \text{ m s}^{-2}$  is the effective gravitational

acceleration. The dynamic viscosity  $\mu = 1.3 \times 10^{-3} \text{ kg m}^{-1} \text{ s}^{-1}$  for pure water at  $10^\circ\text{C}$ , and is within a few percent of this value for seawater.

### 1.6 Salt Conservation

Like the total fluid mass, the dissolved salts in seawater can be neither created nor destroyed, provided that biological fluxes and other changes of state can be neglected, as they will be here. This means that the concentration of dissolved salts, or salinity,  $S$  of a fluid parcel can change only by diffusion:

$$\frac{DS}{Dt} = k_S \nabla^2 S. \quad (1.12)$$

Detailed consideration of thermodynamics shows that thermal and pressure gradients can also cause salt diffusion; however, these effects are small and are neglected here. A characteristic value for the saline diffusivity  $k_S$ , which depends on temperature, is  $k_S = 1.5 \times 10^{-9} \text{ m}^2 \text{ s}^{-1}$ . For a 0.125M saline solution diffusing into pure water at temperatures near 300 K, an empirical expression is  $k_S = k_1 \exp(-k_2/T)$ , with  $k_1 = 4.20 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$  and  $k_2 = 2368 \text{ K}$ . Oceanographic salinity values are now generally given in the dimensionless units of the practical salinity scale, which are nearly equivalent to parts per thousand by mass.

### 1.7 Thermodynamic Energy Balance

The first law of thermodynamics relates the change in internal energy of a thermodynamic system to the work done and the applied heating. For a compressible fluid, this principle can be expressed in terms of the rate of change following the motion of the internal energy  $\hat{e}$  and density  $\rho$  (or equivalently, specific volume  $1/\rho$ ) of a fluid parcel:

$$\rho \left[ \frac{D\hat{e}}{Dt} + p \frac{D}{Dt} \left( \frac{1}{\rho} \right) \right] = k_T \nabla^2 T + \chi + \rho Q_e. \quad (1.13)$$

Here  $k_T$  is the thermal conductivity,  $\chi = 2\mu [\Sigma_{ij} e_{ij} e_{ij} - (\nabla \cdot \mathbf{u})^2/3]$  is internal mechanical heating from viscous dissipation, and  $Q_e$  is external heating. For pure water at  $10^\circ\text{C}$ ,  $k_T = 5.8 \times 10^{-1} \text{ J m}^{-1} \text{ s}^{-1} \text{ K}^{-1}$ . A term representing the chemical potential for salinity could be included in (1.13) but is generally small and is neglected here.

### 1.8 Equation of State

The physical properties of seawater in thermodynamic equilibrium are described by an equation of state that relates one thermodynamic variable, usually the density  $\rho$  or