The Black–Scholes Model

The Black–Scholes option pricing model is the first, and by far the best-known, continuous-time mathematical model used in mathematical finance. Here, it provides a sufficiently complex, yet tractable, testbed for exploring the basic methodology of option pricing.

The discussion of extended markets, the careful attention paid to the requirements for admissible trading strategies, the development of pricing formulae for many widely traded instruments and the additional complications offered by multi-stock models will appeal to a wide class of instructors. Students, practitioners and researchers alike will benefit from the book's rigorous, but unfussy, approach to technical issues. It highlights potential pitfalls, gives clear motivation for results and techniques, and includes carefully chosen examples and exercises, all of which makes it suitable for self-study.

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Preface

The development of modern financial markets can be traced back to two events in the USA in 1973, both of which revolutionised market practice, for very different reasons. One of these revolutions was essentially institutional: the opening of the world's first options exchange in Chicago allowed options to be exchanged in much the same way as stocks (that is, through a regulated exchange) rather than having to be traded 'over the counter' as separate contracts between buyer and seller. The second upheaval was purely theoretical: the publication in the *Journal of Political Economy* of the now famous paper by Fischer Black and Myron Scholes (extended by Robert Merton in the same year), which developed arbitrage techniques for pricing and hedging options, and presented the now ubiquitous Black–Scholes formula for the rational pricing of European call options.

By the late 1970s the basis of their arguments, and the link with martingale theory in particular, had become well enough understood to allow the rapid development of this theoretical breakthrough, which has, since the 1980s, pre-occupied a host of financial economists and mathematicians (principally probabilists) and has given rise to the new profession of quantitative analyst (or 'quant'), which has attracted into the finance sector a large section of the best graduates with mathematics, physics, statistics or computer science degrees. This, in turn, has spawned a host of postgraduate courses emphasising market practice and taught in business schools, but increasingly also courses attached to mathematical sciences departments, focusing on the underlying mathematics, much of which is of comparatively recent origin.

At the same time, finance practitioners have led the explosive, largely unregulated, growth of new financial instruments, grouped together under the term 'derivative securities', which are constantly being devised to meet (or create) demand for specific tailor-made financial products in the banking, currency, insurance, energy and mortgage markets. Hedge funds, which specialise in trading these highly leveraged products, involving huge sums, have become major players in most developed economies. While the mathematical theory underlying their activities is based firmly on market models that exclude arbitrage opportunities (colloquially, a 'free lunch'), in practice much of the motivation comes from the search for risk-free profits, or, perhaps more accurately, the exploitation of market imperfections which briefly create highly marginal profits available through rapid, large-scale trading. This leads to secondary markets whose size overshadows global primary trade – by 2007 the annual volume of derivative trades had reached one quadrillion (10^{15}) US dollars, ten times the global industrial output over the past century – and where incautious, sometimes politically driven, decisions can leave banking institutions exposed to colossal losses, as was demonstrated painfully by the global banking crises of 2008–9 that continue to haunt the global economy.

All this suggests that a more thorough understanding of the principles underlying market practice is essential both for the improvement of that practice and for its regulation. Like nuclear power or the combustion engine, modern financial markets cannot be un-invented; instead, clear insight into their purpose, workings and potential benefits, which necessarily involves mastery of their mathematical basis, is a pre-requisite for adjusting market practice and preventing its abuse.

We will focus attention on the development of the Black–Scholes pricing model and its ramifications. Unlike its much simpler discrete-time binomial counterpart, the Cox–Ross–Rubinstein model (see [DMFM]), a proper understanding of this model requires substantial mathematical tools, principally from the stochastic calculus, which are developed carefully in [SCF]. The random dynamics of stock prices in the Black–Scholes model are based upon the Wiener process (often called Brownian motion). Despite its greater mathematical complexity, the continuous-time model produces a unique pricing formula for vanilla European options which is simpler than its discrete-time counterpart (the CRR formula described in [DMFM]), and has been universally adopted as a standard tool by finance practitioners.

Chapters 1–3 present the basic single-stock model for a general European derivative, with a focus on the explicit formulae for pricing calls and puts, and give a careful account of restrictions on admissible trading strategies. Since arbitrage opportunities usually involve trading in derivatives, the assets held in such strategies should include holdings in the derivatives being priced, and we show that, in our model, the prices of derivative and the replicating strategy must coincide if arbitrage is to be avoided. Option prices are derived in detail for vanilla European options and the unique admissible replicating strategy is constructed and related to the Black–Scholes PDE and to sensitivity measures for the option price relative to its parameters. The key roles of the risk-neutral probability and the representation of martingales by stochastic integrals are highlighted.

Chapter 4 extends and applies the Black–Scholes model in a variety of settings: options on foreign currencies, on futures and on other options. A structural model of credit risk is shown to fit the option pricing setting, a pricing model with time-dependent parameters is introduced, American call options are considered briefly, and the chapter closes with a description of the growth-optimal portfolio. Chapter 5 extends the discussion to the more exotic barriers, lookbacks and Asian options. A two-asset Black–Scholes model is first considered in Chapter 6 before presenting a general multi-asset pricing model, requiring more general versions of the Lévy and Girsanov theorems.

We restrict ourselves to the Black–Scholes setting and its immediate generalisations throughout this volume, working with the natural filtration of a given Wiener process and keeping our reliance on general martingale theory to a minimum. Notable features include the justification of derivative prices by means of replicating strategies and the care taken at the outset in defining the class of admissible trading strategies. The emphasis is on honest proofs of the results we discuss, with much attention given to specific examples and calculation of pricing formulae for different types of options. As usual, the many exercises, whose solutions are made available on the linked website www.cambridge.org/9781107001695, form an integral part of the development of the theory and applications.

We wish to thank all who have read the drafts and provided us with feedback.