Phase Transitions in Random Planar Graphs

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Abstract

We discuss classical results on phase transitions in Erdős–Rényi random graphs and related recent results on random planar graphs and random graphs embeddable on other orientable surfaces of constant genus. The main focus is on how imposing the planarity constraint (or more generally, the genus constraint) affects the global and local structure of random graphs, such as their component structure, local limits, and maximum degree.

1 Introduction

Erdős and Rényi laid the foundations for the theory of random graphs in their seminal work [43, 44]. Since then, various random graph models, at the intersection of different disciplines, including combinatorics, discrete probability, computer science, statistical physics, and network sciences, have been introduced and extensively studied. By now there are several monographs devoted solely to random graphs, e.g., [24, 47, 57, 101, 102].

In the context of random graphs, phase transitions were first discussed by Erdős and Rényi, who discovered in [44] that the random graph undergoes a drastic change in the size and structure of its components. Typically there are many 'small' components – each of which contains at most one cycle – if the average degree is smaller than one, while there is a 'giant' component (that is, a component whose size is much larger than the size of any other component) – which contains at least two cycles – if the average degree is larger than one. Erdős and Rényi showed in [44] that in the 'evolution' of a uniform random graph with n vertices and $N(n)$ edges, the typical size of the largest component changes from logarithmic order $\Theta(\ln n)$ to sublinear order $\Theta(n^{2/3})$ and then to linear order $\Theta(n)$, when the average degree increases from a constant smaller than one, to a constant equal to one, and then to a constant larger than one. Erdős and Rényi wrote in [44]:

> *'This double "jump" of the size of the largest component when $N(n)/n$ passes the value 1/2 is one of the most striking facts concerning random graphs'.*

This suggests the existence of a *'discontinuous' phase transition* in the evolution of the random graph when the average degree passes through the critical value one. Bollobás [23] and Łuczak [73] however showed that the Erdős–Rényi random graph undergoes not a discontinuous, but a *continuous* phase transition.

These fascinating results have been generalised to various other random graph models and strengthened in several ways. One random graph model that has received considerable attention in recent years is a random planar graph, whose study was pioneered by McDiarmid, Steger, and Welsh [77]. Imposing the planarity constraint on random graphs typically makes their investigation more challenging, but many exciting results have been obtained. It is well known that in the very sparse regime when the average degree is smaller than one, a random planar graph typically behaves similarly to the Erdős–Rényi random graph, because typically a very sparse Erdős–Rényi random graph is planar. However, different properties have been found to emerge when the average degree is larger than one. For example, Łuczak and Kang showed in [59] that surprisingly there are *two critical periods* in the evolution of a random planar graph, in stark contrast to the Erdős–Rényi random graph, which features only a *single* critical period. In a random planar graph, the *first critical point* is when the average degree equals *one*, where the unique giant component emerges, analogously to the Erdős–Rényi random graph, but the asymptotic size of the giant component in a random planar graph is roughly half the asymptotic size of the giant component in the Erdős–Rényi random graph. The *second critical point* that a random planar graph exhibits is when the average degree equals *two*, where the giant component contains almost all vertices. Such similarities and differences observed in phase transitions in the Erdős–Rényi random graph and a random planar graph form the basis of this paper.

Starting from the work by McDiarmid [75], many results on random planar graphs have now been extended to random graphs embeddable on other orientable surfaces of constant genus. It is well known that random graphs on orientable surfaces of constant genus enjoy *universal properties*. Such properties, especially in terms of component structures, local weak limits, and the maximum degree, will be discussed in Sections 3, 4, and 5, respectively.

2 Preliminaries

2.1 Erdős–Rényi random graphs

The most popular random graph model is arguably the binomial random graph $G_{n,p}$, i.e., a graph with vertex set $[n] := \{1, \ldots, n\}$, where each unordered pair of distinct vertices is joined by an edge independently with probability $p \in [0, 1]$. Another well-studied model is the uniform random graph introduced by Erdős–Rényi in [43]. To be more precise, let $\mathcal{G}(n, m)$ denote the set of all (vertex-labelled simple undirected) graphs with vertex set $[n]$ and with exactly m edges and let $G(n, m)$ be a graph chosen uniformly at random from $\mathcal{G}(n, m)$. Note that $G_{n,p}$ is essentially equivalent to $G(n, m)$ when $\binom{n}{2}p \sim m$ (see e.g., [47, Section 1.1] or [57, Section 1.4]).

Throughout the paper we let $G(n, m) \in_u \mathcal{G}(n, m)$ denote a graph chosen uniformly at random from $\mathcal{G}(n, m)$ and call it the Erdős–Rényi random graph $G(n, m)$.

2.2 Random planar graphs

The extensive study of planar graphs, planar maps (i.e., graphs that are *embedded* in the plane), and related families of graphs goes back to Tutte [100], leading to a plethora of impressive results, including results about asymptotic enumeration and bijections [9, 10, 14, 15, 26, 27, 29, 38, 41, 58, 72, 94, 99, 100], scaling and local limits [1, 4, 63, 66, 67, 68, 69, 70, 71, 78, 97, 98], random sampling and generation [13, 16, 17, 18, 48, 89, 95], and structural properties [12, 19, 28, 32, 33, 34, 36, 39, 40, 42, 46, 49, 50, 51, 52, 53, 59, 60, 61, 62, 63, 64, 65, 75, 76, 77, 80, 81, 82, 83, 84, 85, 86]. In particular, Noy [80] provided an excellent survey on the enumeration of planar graphs and asymptotic properties of random planar graphs. Many of these results are largely based on generating functions associated with decompositions of graphs into parts with appropriate connectivity properties and the methods from analytic combinatorics, whose general framework and theory can be found in [37] and [45].

In view of the main themes of the paper, which are the component structure, local limits, and the maximum degree of a random graph embeddable on an orientable surface of constant genus, the most important results concerning the component structure are obtained in [27, 46, 53, 59, 61, 64, 75, 77], the local limits in [63, 97], and the maximum degree in [34, 36, 39, 40, 60, 76, 86].

These intensive studies revealed rich and complex behaviour of random graphs on orientable surfaces. But, most of them dealt with random graph models *without a specification of the number of edges*, therefore not providing evolutionary viewpoints, or with random graph models in *dense regimes* corresponding to *late stable stages* in the evolution of random graphs.

The primary topic of this paper is *phase transition phenomena* in *sparse* random planar graphs (or more generally, *sparse* random graphs on orientable surfaces), which are unveiled only if the number of edges is taken into account in the model and if it is not too large, so that *early unstable* stages in the evolution of random graphs are properly detected, especially in the light of the phase transition phenomena discovered by Erdős and Rényi in [44].

Given $g \in \mathbf{N} \cup \{0\}$, let $\mathbb{S}_g$ denote the orientable surface of genus g. A graph H is called *embeddable* on $\mathbb{S}_g$ if H can be drawn on $\mathbb{S}_g$ without crossing edges. A graph embeddable on $\mathbb{S}_g$ will simply be called a graph on $\mathbb{S}_g$. We let $\mathcal{S}_g(n,m)$ denote the set of all (vertex-labelled simple undirected) graphs on $\mathbb{S}_g$ with vertex set $[n]$ and with exactly m edges. Note that the case when the genus $g = 0$ corresponds to planar graphs and

$$\mathcal{S}_0(n,m) \;\subset\; \cdots \;\subset\; \mathcal{S}_g(n,m) \;\subset\; \mathcal{S}_{g+1}(n,m) \;\subset\; \cdots \;\subset\; \mathcal{G}(n,m).$$

Throughout the paper we assume $g \in \mathbf{N} \cup \{0\}$ is a fixed constant. We let $S_g(n,m) \in_u \mathcal{S}_g(n,m)$ denote a graph chosen uniformly at random from $\mathcal{S}_g(n,m)$ and call it the random graph $S_g(n,m)$ on $\mathbb{S}_g$.

2.3 Local weak limit

The concept of *local weak convergence* was formally introduced by Benjamini and Schramm [11] and independently by Aldous and Steele [3]. For an overview of the subject, see the book of van der Hofstad [102, Chapter 2].

A (possibly infinite) graph H is *locally finite* if every vertex of H has finite degree. A *rooted graph* is a pair (H,r), where H is a locally finite graph and r is a vertex of H. Two rooted graphs (H_1,r_1) and (H_2,r_2) are *isomorphic*, which is denoted by

$$(H_1,r_1) \;\cong\; (H_2,r_2),$$

if there exists an isomorphism from H_1 to H_2 that takes r_1 to r_2. Given $\ell \in \mathbf{N}$ and a rooted graph (H,r), the ball $B_\ell(H,r)$ of radius ℓ in H with centre r is the subgraph of H induced on the vertices with distance at most ℓ from r and it is considered as a rooted graph with root r.

Given two rooted graphs (H_1, r_1) and (H_2, r_2), we define the distance between them by

$$d\Big((H_1, r_1), (H_2, r_2)\Big) \;:=\; 2^{-k},$$

where k is the supremum of all $\ell \in \mathbf{N}$ satisfying

$$B_\ell(H_1, r_1) \cong B_\ell(H_2, r_2).$$

It is not difficult to verify that d is a metric on the space $\mathcal{G}$ of isomorphism classes of *connected* rooted graphs. Using the fact that the law of a random connected rooted graph is a probability measure on $\mathcal{G}$, we can define a 'limiting graph' for a sequence $\big((G_n, r_n)\big)_{n\in\mathbf{N}}$ of random connected rooted graphs (G_n, r_n) as follows. A random connected rooted graph (G_*, r_*) is the (Benjamini–Schramm) *local weak limit* of (G_n, r_n), which is denoted by

$$(G_n, r_n) \quad \xrightarrow{d} \quad (G_*, r_*),$$

if the law of (G_n, r_n) converges weakly to the law of (G_*, r_*). This is equivalent to

$$\mathbb{P}\left[B_\ell\left(G_n, r_n\right) \cong B_\ell(H, r_H)\right] \quad \to \quad \mathbb{P}\left[B_\ell(G_*, r_*) \cong B_\ell(H, r_H)\right] \quad \text{as} \quad n \to \infty$$

for each $\ell \in \mathbf{N}$ and each fixed rooted graph (H, r_H).

Roughly speaking, the local weak limit of (G_n, r_n) depends only on the local structure of G_n around the root r_n. For not necessarily connected rooted graphs (G_n, r_n) we define the local weak limit as the local weak limit of the connected rooted graphs (G_{r_n}, r_n) where G_{r_n} denotes the component of G_n that contains the root r_n.

Given a constant $a \in [0, 1]$ and two random rooted graphs (H_1, r_1) and (H_2, r_2), we denote by

$$a \cdot (H_1, r_1) \,+\, (1-a) \cdot (H_2, r_2)$$

the random rooted graph (G, r) that satisfies

$$\mathbb{P}\left[(G, r) \cong (H, r_H)\right] \;=\; a \cdot \mathbb{P}\left[(H_1, r_1) \cong (H, r_H)\right] + (1-a) \cdot \mathbb{P}\left[(H_2, r_2) \cong (H, r_H)\right]$$

for each fixed rooted graph (H, r_H).

2.4 Notations and notions

Throughout the paper all asymptotics are taken as $n \to \infty$. We say a property holds *with high probability*, or *whp* for short, if it holds with probability tending to one as $n \to \infty$. In addition to the standard Landau notation we use the following asymptotic notions. Given a sequence $(X_n)_{n\in\mathbf{N}}$ of random variables and a function $f \colon \mathbf{N} \to \mathbf{R}_{\geq 0}$, we say that

- $X_n = O_p(f(n))$, if for every $\delta > 0$, there exist $C_0 = C_0(\delta) > 0$ and $N_\delta \in \mathbf{N}$ such that with probability at least $1 - \delta$, we have $|X_n| \leq C_0 f(n)$ for every $n \geq N_\delta$;
- $X_n = \Theta_p(f(n))$, if for every $\delta > 0$, there exist $C_1 = C_1(\delta) > 0, C_2 = C_2(\delta) > 0$, and $N_\delta \in \mathbf{N}$ such that with probability at least $1-\delta$, we have $C_1 f(n) \leq |X_n| \leq C_2 f(n)$ for every $n \geq N_\delta$.

Given a graph H with vertex set $[n]$ and with m edges, we call

$$\frac{2m}{n} = \frac{1}{n}\sum\nolimits_{v\in[n]} d(v)$$

the *average degree* of H. We denote by $\Delta(H)$ the *maximum degree* of H. A *component* of H is a maximal connected subgraph of H. Given a component $C(H)$ of H, its *size* $|C(H)|$ is the number of vertices in $C(H)$. For $i \in \mathbf{N}$, let $L_i(H)$ denote the i-th largest component of H. A graph $R(H) := H \setminus L_1(H)$, which is a subgraph of H obtained by deleting $L_1(H)$, is called the *fragment* of H. A component of H is called a *tree* if it contains no cycle, *unicyclic* if it contains exactly one cycle, and *complex* if it contains at least two cycles, respectively.

A *bare* path is a path whose internal vertices all have degree exactly two. Given a graph H, the *2-core* of H is the maximal subgraph of H whose minimum degree is at least two, and the *kernel* of H is a multigraph obtained from the 2-core of H by replacing each bare path by an edge.

The genus $g = g(H)$ of a graph H is defined as the minimum number of handles that must be attached to a sphere in order to be able to embed the graph H without any crossing edges.

3 Component structure

Among many features of the Erdős–Rényi random graph $G(n, m)$, the most prominent one is the *phase transition phenomenon* in the size and structure of its components, when the average degree $2m/n$ goes through the critical value one. At this point the graph typically changes in structure from being *planar* with relatively small components (of logarithmic size) to being *non-planar* with a unique giant component (of linear size). This result became a benchmark in the theory of random graphs.

In this section we will first overview the most important results in this regard. We then continue with corresponding results on the random graph $S_g(n, m)$ on $\mathbb{S}_g$.

3.1 Component structure in the Erdős–Rényi random graph

We begin with the classic results on the phase transition in the component structure of $G(n, m)$ that Erdős and Rényi discovered in [44]. (Some assertions in [44] relied on flawed arguments, but they were subsequently proved and improved by other authors – for a summary of the history, see e.g., [55, the two paragraphs before Theorem 8].) The following results are presented in a modified form to use unified notation throughout the paper for easy comparison.

Theorem 3.1 *Let $G = G(n, m) \in_u \mathcal{G}(n, m)$. Assume $2m/n$ tends to a constant $c \in [0, \infty)$.*

(a) If $c < 1$ (in the subcritical phase*), then whp every component of G is either a tree or unicyclic. In addition, whp G is planar and*

$$|L_1(G)| = O(\ln n).$$

(b) If $c = 1$ *(at the* critical point*), then*

$$|L_1(G)| \;=\; \Theta_p(n^{2/3}).$$

The probability that G *is planar is bounded away from 0 and 1.*

(c) If $c > 1$ *(in the* supercritical phase*), then whp* G *is not planar and*

$$|L_1(G)| \;=\; \Theta(n).$$

In short, the Erdős–Rényi random graph $G(n, m)$ undergoes a phase transition when the average degree is equal to one. This is indeed not surprising, in view of the *universality principle* in percolation theory, which says that the critical threshold for the emergence of the giant cluster is determined by the local structure of the graph, for example, the expected degree of a random vertex being equal to one.

In fact, Theorem 3.1 (b) holds only when $2m/n$ tends to the critical value one 'fast enough'. Bollobás [23] and Łuczak [73] strengthened Theorem 3.1 (b) when the average degree $2m/n$ is 'close' to 1. One of the main difficulties is to quantify how fast $2m/n$ converges to 1, in order to be able to capture meaningful changes in the size and structure of components. Bollobás [23] and Łuczak [73] found appropriate parametrisations of the average degree, determining the correct width of the critical window to be $O(n^{2/3})$. In addition to the sizes of components, they also studied whether these components are *trees*, *unicyclic*, or *complex*. Their results are further strengthened, e.g., by Janson, Knuth, Łuczak, and Pittel [55]. Revealing only the tip of the iceberg, we oversimplify their results as follows.

Theorem 3.2 *Let* $G = G(n, m) \in_u \mathcal{G}(n, m)$ *and* $R(G) := G \setminus L_1(G)$. *Assume*

$$2m/n \;=\; 1 + \epsilon \quad \textit{for} \quad \epsilon = \epsilon(n) = o(1).$$

(a) If $\epsilon n^{1/3} \to -\infty$ *(in the* weakly subcritical phase*), then whp every component of* G *is either a tree or unicyclic (and thus whp* G *is planar). For every* $i \in \mathbf{N}$, *whp* $L_i(G)$ *is a tree and*

$$|L_i(G)| \;=\; (1 + o(1))\, 2|\epsilon|^{-2} \ln(|\epsilon|^3 n).$$

(b) If $\epsilon n^{1/3} \to a$ *for a constant* $a \in \mathbf{R}$ *(in the* critical phase*), then the probability that* G *has complex components is bounded away from 0 and 1. In addition, the probability that* G *is planar is bounded away from 0 and 1. For every* $i \in \mathbf{N}$,

$$|L_i(G)| \;=\; \Theta_p(n^{2/3}).$$

(c) If $\epsilon n^{1/3} \to +\infty$ *(in the* weakly supercritical phase*), then whp* G *is not planar,* $L_1(G)$ *is complex, and*

$$|L_1(G)| \;=\; (1 + o(1))\, 2\epsilon n.$$

In addition, whp every component of $R(G)$ *is either a tree or unicyclic. For* $i \geq 2$, *whp* $L_i(G)$ *is a tree and*

$$|L_i(G)| \;=\; (1 + o(1))\, 2\epsilon^{-2} \ln(\epsilon^3 n).$$

The behaviour of the *critical* Erdős–Rényi random graph $G(n,m)$ is more complex than as stated in Theorem 3.2 (b). For example, Aldous [2] showed that the ordered scaled sizes of the largest components converge to a multiplicative coalescent process, which is equivalent to ordered excursion lengths of the associated Brownian excursions.

The genus of a graph is one of the most fundamental topological characteristics of a graph, and thus it is intriguing to investigate the genus of a random graph. In Theorem 3.1 and Theorem 3.2 we discussed the planarity of the Erdős–Rényi random graph. In the *critical phase* we have stronger results than Theorem 3.2 (b). Łuczak, Pittel, and Wierman [74] showed that there exists a function $F : \mathbf{R} \to (0,1)$, $a \mapsto F(a)$, such that if $m = n/2 + a\,n^{2/3}$ for a constant $a \in \mathbf{R}$, then

$$\lim_{n\to\infty} \mathbb{P}[G(n,m) \text{ is planar}] \;=\; F(a), \tag{3.1}$$

where $\lim_{a\to-\infty} F(a) = 1$ and $\lim_{a\to+\infty} F(a) = 0$. Noy, Ravelomanana, and Rué [82] derived the analytic expression for $F(a)$.

The genus of the Erdős–Rényi random graph $G(n,m)$, or more precisely, the genus of $G_{n,p}$, was first studied by Archdeacon and Grable [6], who showed among other results that whp the genus of $G_{n,p}$ is $(1+o(1))pn^2/12$, if $p^2(1-p^2) \geq 8(\ln n)^4/n$. As noted in [6], results for the genus of $G_{n,p}$ can be transferred into analogous results for the genus of $G = G(n,m)$. Rödl and Thomas considered dense cases when m is superlinear in n and showed in [92] that the typical genus of G is superlinear in n. More precisely, whp $g(G) = (1+o(1))\,m/6$ when $m = \Theta(n^2)$ and $g(G) = (1+o(1))\,k\,m/(2k+4)$ when $m = \omega(n^{(k+2)/(k+1)})$ but $m = o(n^{(k+1)/k})$ for $k \in \mathbb{N}$. Complementing these results, Dowden, Kang, and Krivelevich [35] determined the genus of G in sparse regimes. For example, if $2m/n = 1+\epsilon$ for $\epsilon = \epsilon(n) = o(1)$ and $\epsilon n^{1/3} \to +\infty$ (in the *weakly supercritical phase*), then whp

$$g(G) \;=\; (1+o(1))\epsilon^3 n/3, \tag{3.2}$$

which is roughly one quarter of the asymptotic size of the *kernel* of G.

What else can we say about the genus $g(G)$ of $G = G(n,m)$ near the critical point? We might perhaps be able to guess the behaviour of $g(G)$ from what we already know about the genus in the weakly subcritical and supercritical phases. To this end, assume $2m/n = 1+\epsilon$ for $\epsilon = \epsilon(n) = o(1)$. Note that Theorem 3.2 (a) says that if $\epsilon n^{1/3} \to -\infty$ (in the *weakly subcritical phase*), then whp $g(G) = 0$. On the other hand, (3.2) says that if $\epsilon n^{1/3} \to +\infty$ (in the *weakly supercritical phase*), then whp $g(G) = \Theta(\epsilon^3 n) = \omega(1)$. Summing up, $g(G)$ changes from zero to $\omega(1)$, as $\epsilon n^{1/3}$ increases from $-\infty$ to $+\infty$ through the critical phase. A natural question is whether we can describe this change more precisely. In view of the planarity result (3.1) by Łuczak, Pittel, and Wierman [74], we are interested in the analogous result for a fixed constant $g \in \mathbf{N}$ and are curious whether there exists a function $F_g : \mathbf{R} \to (0,1)$, $a \mapsto F_g(a)$, such that if $m = n/2 + a n^{2/3}$ for a constant $a \in \mathbf{R}$, then

$$\lim_{n\to\infty} \mathbb{P}[G(n,m) \text{ has genus } g] \;=\; F_g(a).$$

If such a function F_g exists, it would be interesting to determine the behaviour of F_g. For example, is it continuous and increasing?

3.2 Component structure in a random planar graph

Early work on random planar graphs, or more generally, random graphs on surfaces, has been restricted to random graph models *without a specification of the number of edges*, but more recent work includes the study of random graphs on surfaces with given numbers of vertices and edges. Below we discuss the most important results about the component structure in random graph models both with and without a specification of the number of edges.

Given $g \in \mathbf{N} \cup \{0\}$, let $\mathcal{S}_g(n)$ denote the set of all (vertex-labelled simple undirected) graphs on $\mathbb{S}_g$ with vertex set $[n]$ and let $S_g(n) \in_u \mathcal{S}_g(n)$ be a random graph which is chosen uniformly at random from $\mathcal{S}_g(n)$. Note that Euler's formula implies that if $m > 3n - 6 + 6g$, then $\mathcal{S}_g(n, m) = \emptyset$. Thus we have

$$\mathcal{S}_g(n) \ = \bigcup_{0 \le m \le 3n-6+6g} \mathcal{S}_g(n, m).$$

When $2m/n$ tends to a constant $c \in [0, 1)$, it follows from Theorem 3.1 (a) that whp $G(n, m)$ is planar, implying that

$$\frac{|\mathcal{S}_0(n, m)|}{|\mathcal{G}(n, m)|} \ \to \ 1 \quad \text{if} \quad 2m/n \ \to \ c \in [0, 1)$$

and thus any property that holds whp in $G(n, m)$ will also hold whp in the random graph $S_g(n, m)$ on $\mathbb{S}_g$ for every $g \in \mathbf{N} \cup \{0\}$. As a consequence, we have, e.g., that whp

$$|L_1(S_g(n, m))| \ = \ O(\ln n) \quad \text{if} \quad 2m/n \ \to \ c \in [0, 1).$$

The asymptotic behaviour of $|\mathcal{S}_0(n)|$ and of $|\mathcal{S}_0(n, m)|$ when $2m/n = c$ for a constant $c \in (2, 6)$ were determined by Giménez and Noy [53], using a sophisticated singularity analysis of generating functions, based on prior work by Bender, Gao, and Wormald [10]. This work was extended by Bender and Gao [9] and independently by Chapuy, Fusy, Giménez, Mohar, and Noy [27] to arbitrary constant $g \in \mathbf{N}$. Among other interesting results, Chapuy, Fusy, Giménez, Mohar, and Noy showed in [27] that whp the size of the largest component in $S_g(n)$ is $n - O(1)$ and that the number of edges in $S_g(n)$ is asymptotically normally distributed with mean αn and variance βn, where $\alpha \approx 2.2133$ and $\beta \approx 0.4303$ are constants (which are independent of g). Their results imply that the average degree of $S_g(n)$ is very close to 4.42 and that $S_g(n)$ and $S_g(n, m)$ with $2m/n = c$ for a constant $c \in (2, 6)$ behave like a *dense* Erdős–Rényi random graph $G(n, m)$ where the giant component containing nearly all but a bounded number of vertices appears only when $2m/n$ is close to the connectivity threshold, i.e., when $m \sim n \ln n$.

Thus, when studying the random graph $S_g(n, m)$ on $\mathbb{S}_g$ in the light of the phase transition phenomenon in the size and structure of components in the Erdős–Rényi random graph $G(n, m)$, the most interesting range to focus on is the *sparse* regime when the average degree $2m/n$ tends to a constant $c \in [1, 2]$. Łuczak and Kang [59] showed that such a sparse random planar graph undergoes *two critical phases*, in comparison to the single critical phase that the Erdős–Rényi random graph undergoes. In the random planar graph $S_0 = S_0(n, m)$, the first critical point $c = 1$ is when the unique giant component $L_1(S_0)$ emerges, and the second critical point $c = 2$ is when $L_1(S_0)$ covers $n - \Theta(n^{3/5})$ vertices. In the *intercritical phase* when $c \in (1, 2)$, whp $|L_1(S_0)| \sim (c-1)n$. This work has been extended by Kang, Moßhammer, and Sprüssel [64] to $S_g(n, m)$ for any arbitrary constant $g \in \mathbf{N} \cup \{0\}$. Below we discuss these results more precisely.

Analogously to the Erdős–Rényi random graph $G(n, m)$ (see Theorem 3.2), let us look closer into the *first critical phase* of the random graph $S_g(n, m)$ on $\mathbb{S}_g$. Łuczak and Kang [59] (for $g = 0$) and Kang, Moßhammer, and Sprüssel [64] (for arbitrary constant $g \in \mathbf{N} \cup \{0\}$) determined the correct width of the first critical window to be $O(n^{2/3})$ and investigated the size and structure of the largest components. Among other results, they proved the following.

Theorem 3.3 *Let $g \in \mathbf{N} \cup \{0\}$. Let $S_g = S_g(n, m) \in_u \mathcal{S}_g(n, m)$ and $R(S_g) := S_g \setminus L_1(S_g)$. Assume*

$$2m/n \;=\; 1 + \epsilon \quad \textit{for} \quad \epsilon = \epsilon(n) = o(1).$$

(a) If $\epsilon n^{1/3} \to -\infty$ (in the first weakly subcritical phase*), then for every $i \in \mathbf{N}$, whp $L_i(S_g)$ is a tree and*

$$|L_i(S_g)| \;=\; (1 + o(1))\, 2|\epsilon|^{-2} \ln(|\epsilon|^3 n).$$

(b) If $\epsilon n^{1/3} \to a$ for a constant $a \in \mathbf{R}$ (in the first critical phase*), then the probability that S_g contains a complex component is bounded away from 0 and 1. For every $i \in \mathbf{N}$,*

$$|L_i(S_g)| \;=\; \Theta_p(n^{2/3}).$$

(c) If $\epsilon n^{1/3} \to +\infty$ (in the first weakly supercritical phase*), then whp $L_1(S_g)$ has genus g and it is complex. Furthermore,*

$$|L_1(S_g)| \;=\; (1 + o(1))\epsilon n + O_p(n^{2/3}).$$

In addition, whp $R(S_g)$ is planar and it has $O_p(1)$ many complex components, each of which has size $O_p(n^{2/3})$. For $i \geq 2$,

$$|L_i(S_g)| \;=\; \Theta_p(n^{2/3}).$$

Theorem 3.3 (a) and (b) follow from Theorem 3.2 (a) and (b), respectively. But Theorem 3.3 (c) requires quite involved counting arguments based on the *core-kernel approach*, which will be discussed in Section 3.4.

It is worth mentioning that in the first weakly supercritical phase in $S_g(n, m)$, whp the genus of the giant component is g, while the genus of the fragment is zero, in other words, typically the giant component contains the full information on the genus of the whole graph.

From the first parts of Theorem 3.3 (c) and Theorem 3.2 (c) we have that the asymptotic size of the giant component in $S_g(n, m)$ is about ϵn, while the size of the giant component in $G(n, m)$ is about $2\epsilon n$.

The second parts of Theorem 3.3 (c) and Theorem 3.2 (c) tell us that whp the fragment in $S_g(n, m)$ contains (a bounded number of) *complex* components, but whp the fragment in $G(n,m)$ does not contain a complex component. In addition, for $i \geq 2$, the size of the i-th largest component in $S_g(n,m)$ is $\Theta_p(n^{2/3})$, while the counterpart in $G(n,m)$ is of order $\Theta(\epsilon^{-2}\ln(\epsilon^3 n)) = o(n^{2/3})$.

Summing up, Theorem 3.3 (c) clearly shows that in the first weakly supercritical phase, (the size and structure of) the largest components of $S_g(n,m)$ behave quite differently from those of $G(n,m)$. Roughly speaking, imposing the planarity or the genus constraint on a random graph makes its edges spread *more evenly* over different components, in comparison to the 'Rich-Get-Richer' principle that $G(n,m)$ enjoys. As a consequence, the asymptotic size of the giant component in $S_g(n,m)$ is only the half the asymptotic size of the giant component in $G(n,m)$, while the asymptotic size of the second largest component in $S_g(n,m)$ is much larger than the counterpart in $G(n,m)$. In addition, typically the fragment of $S_g(n,m)$ contains several components with at least two cycles, but every component in the fragment of $G(n,m)$ contains at most one cycle.

The existence of the *second critical phase* around the critical value two is a special aspect of sparse random planar graphs, or more generally, sparse random graphs on $\mathbb{S}_g$. Needless to say, such a phase does not exist in the Erdős–Rényi random graph.

Based on the phase transition results by Łuczak and Kang [59] (for $g = 0$) and Kang, Moßhammer, and Sprüssel [64] (for arbitrary constant $g \in \mathbf{N} \cup \{0\}$), the next two theorems summarise the results about the size and structure of the giant component $L_1(S_g)$ of $S_g = S_g(n,m)$ and the fragment $R(S_g) = S_g \setminus L_1(S_g)$ in the intercritical and the second critical phases.

In the *intercritical phase*, typically the rescaled asymptotic size of the giant component $L_1(S_g)$ grows *linearly* in $c-1$ and the fragment $R(S_g)$ is planar (cf. [64, Theorem 1.6]).

Theorem 3.4 *Let $g \in \mathbf{N} \cup \{0\}$. Let $S_g = S_g(n,m) \in_u \mathcal{S}_g(n,m)$ and $R(S_g) := S_g \setminus L_1(S_g)$. Assume*

$$2m/n \;\to\; c \in (1,2).$$

Then whp $L_1(S_g)$ has genus g and it is complex. Furthermore,

$$|L_1(S_g)| \;=\; (c-1)n + O_p(n^{2/3}).$$

In addition, whp $R(S_g)$ is planar. For $i \geq 2$,

$$|L_i(S_g)| \;=\; \Theta_p(n^{2/3}).$$

Next we take a closer look into the *second critical phase* when the the average degree $2m/n$ tends to the critical value two. The following result (cf. [64, Theorem 1.5]) indicates that the correct width of the second critical window is $O(n^{3/5})$.

Theorem 3.5 *Let $g \in \mathbf{N} \cup \{0\}$. Let $S_g = S_g(n,m) \in_u \mathcal{S}_g(n,m)$ and $R(S_g) := S_g \setminus L_1(S_g)$. Assume*

$$2m/n \;=\; 2+\eta \quad \text{for} \quad \eta = \eta(n) = o(1).$$

Then whp $L_1(S_g)$ has genus g and it is complex. Furthermore, whp $R(S_g)$ is planar.