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Excerpt
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PART I

Biography

1
The Life of S. S. Shrikhande

Ambat Vijayakumar, an author of this book, writes:

The International Conference on Recent Trends in Graph Theory and Combinatorics (ICRTGC) was held in Cochin, India. It was organized by the Cochin University of Science and Technology, India, during 7–10 June 2010, as a satellite conference of the International Congress of Mathematicians (ICM) 2010 held in Hyderabad, India. The conference logo (Fig. 1.1) was the renowned 'Shrikhande Graph'. I had only a vague memory of having met S. S. Shrikhande in a conference held at the University of Mumbai and I had never heard about his contributions to combinatorial designs, association schemes, and the Shrikhande Graph itself before 2010. Although I do not wish to find excuses for my ignorance, it is surprising that the Shrikhande Graph came to my mind. That prompted me to know more about him and his graph, and from then I was

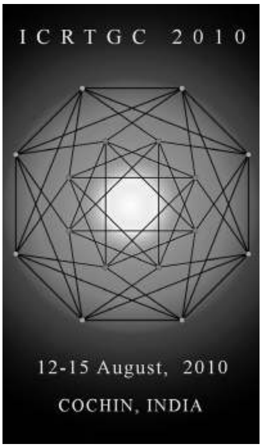

Figure 1.1 Logo of ICRTGC 2010

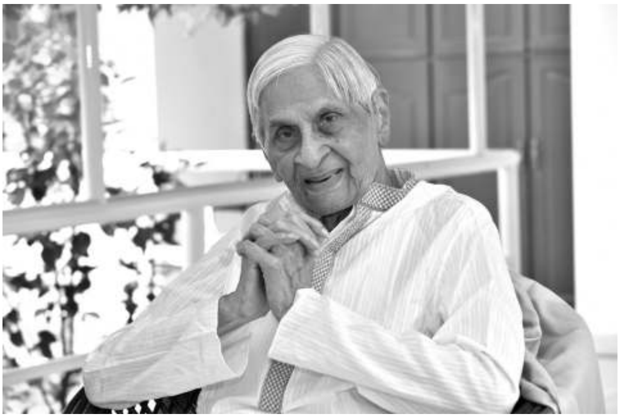

Figure 1.2 Sharadchandra Shankar Shrikhande

madly popularizing the works of Shrikhande to the mathematical community through webinars, articles for students in my mother tongue (Malayalam), and so forth.

This introductory chapter provides a biographical sketch of Sharadchandra Shankar Shrikhande. Shrikhande (Fig. 1.2) was born at Sagar, Madhya Pradesh, on 19 October 1917, as the fifth of ten children to his parents Shankar and Parvathy. Like many other Indian mathematicians, he was not born with a silver spoon; his father being an ordinary worker at a flour mill. He received his basic education at Government High School, Sagar, and passed the school-leaving certificate examination in 1934 with the fifth rank. Consequently, he was awarded a government merit scholarship to continue his studies at Robertson College, Jabalpur, for the intermediate science course, which he completed with flying colours in 1936. This success enabled him to get another scholarship to study the BSc Honours course at the Government College of Science, formerly known as the Institute of Science, Nagpur (since 1906). He completed this course majoring in pure and applied mathematics, with physics as an allied subject, securing the first rank, for which he was awarded the Prakya Ganpat Rao Gold Medal. He was very badly in need of employment, which forced him to move to Calcutta (now Kolkata) in West Bengal, India.

The story of Shrikhande now has to be linked with that of Professor P. C. Mahalanobis (1893–1972), the father of modern Indian statistics and the prestigious Indian Statistical Institute (ISI), which he established in 1930. Though Mahalanobis was a Professor of Physics at the Presidency College, Calcutta, he made remarkable contributions to statistical techniques, in jute sample surveys. Shrikhande joined the ISI in 1940 with financial help, mainly through the King Edward Memorial Fellowship, named after Edward VIII (1894–1972), who was King of the United Kingdom and the Dominions of the British Empire and Emperor of India in 1936. The ISI already had the presence of S. N. Roy, R. C. Bose, K. R. Nair (who later became an FRS), and many others, who all later became doyens in statistics due to their fundamental contributions. He returned to his alma mater at Jabalpur, and later moved to Government College of Science at Nagpur, to work as a lecturer from 1942 to 1958. But he was making regular visits to the ISI to have discussions with R. C. Bose, who introduced him to the theory of statistical designs, Latin squares, and similar topics.

In 1947, on the suggestion of Dr. P. V. Sukhathme (the Statistical Adviser of Indian Council of Agricultural Research, New Delhi), Shrikhande joined the Department of Mathematics and Statistics at the University of North Carolina, USA. In fact, the Government of India had decided to send some Indian scholars to the USA for training under the Public Law 480 – Agricultural Trade Development Assistance Act scheme. That department had just been established, in 1946, under the chairmanship of Professor Harold Hotelling (an American statistician well known for Hotelling's law as well as Hotelling's T-squared distribution) and already had the services of K. A. Bush, R. A. Bradley, R. C. Bose, S. N. Roy, and G. Kallianpur (who later became Director of the ISI).

Shrikhande's formal studies began with a course on Linear Estimation by R. C. Bose. Later he became Bose's first PhD student, and completed his thesis in 1950. Shrikhande describes the return of Bose to the USA from India in 1949 as a 'turning point in my academic career'. K. A. Bush, who later served Washington State University, USA, remarks in *A Survey of Combinatorial Theory* (1973), pp. 79–80: 'Professor Bose set highly imaginative but quite difficult topics for the student's investigation. In the second place, the student had to supply substantial originality. Professor Bose seemed totally unaware of the modern trend where the thesis advisor shows the student how to solve a part of the problem and then suggests that he carry it further with some inconsequential generalization. Instead, Professor Bose expected students to discover appropriate theorems on their own and develop sound techniques for their proof. At this stage, he would read every word, frequently rewrite entire sections to improve the exposition, and sometimes find important refinements or extensions of the results. While this was helpful, the really important element is that he forced the students to do genuine research from the outset. Professor Bose made the students feel that his overwhelming desire was to see each of the students succeed.'

During 1951–53, Shrikhande was an Assistant Professor of Statistics at the University of Kansas, Lawrence, USA, and was Associate Professor at Chapel

Hill during 1958–60. It was during the latter period that the famous refutation of Euler's conjecture on Graeco-Latin squares (which we discuss later in the book) was given by Bose, Shrikhande, and E. T. Parker. Instead of staying on in the USA, Shrikhande returned to India to take up a professorship at Banaras Hindu University, where he worked until 1963. He then joined the University of Bombay as Professor and Head of the Department of Mathematics, where he worked until his formal retirement in 1978. During this period, he was also Director of the Centre for Advanced Study in Mathematics at Bombay. During 1983–86, he was Director of the Mehta Research Institute at Allahabad, which is a research centre for Mathematics and Theoretical Physics (renamed later as Harish Chandra Research Institute).

He had been a Visiting Professor at various US universities, such as the University of Wisconsin, the Ohio State University, State University of New York, Stanford University, and Colorado State University. He has been associated with the Indian Statistical Institute in various capacities. Professor Shrikhande has been a member of a number of reputed societies: Indian National Science Academy, Indian Academy of Sciences, Institute of Mathematical Statistics, and International Statistical Institute, to mention a few.

Shrikhande spent a few years in Nagpur after leaving his position as the Director of the Mehta Research Institute. His last ten years passed very peacefully and were spent in the extremely quiet and serene surroundings of Chinmaya Vijay Ashram, Vijayawada, in Andhra Pradesh. Shrikhande breathed his last on 21 April 2020. Some appreciations are in [88, 89, 94, 96, 97].

Let me conclude this chapter with a few lines about Shrikhande's family. He was married to Shakuntala, who was a school teacher. The couple had four children: Vijay (who passed away in early 1990), Asha (who lives with her husband in Michigan, USA), Mohan (an Emeritus Professor at Central Michigan University), and Anil (who has worked in several companies in the USA and served as the head of some major companies in Delhi).

As a tribute to this great researcher, the discrete mathematics community of India, through the Academy of Discrete Mathematics and Applications (ADMA), instituted the Professor Shrikhande Memorial Lecture, commencing at its twenty-first annual conference held in June 2025 at Cochin University of Science and Technology.

Rosemary Bailey, combinatorialist and statistician, writes:

> On 24 and 25 October 2003, the American Mathematical Society held a meeting at the University of North Carolina in Chapel Hill. A Special Session on *Association Schemes: 1973–2003* ran for the whole meeting, organized by William J. Martin from Worcester Polytechnic Instistute and Dijen K. Ray-Chaudhuri from Ohio State University. Among the speakers were Sharadchandra Shrikhande's son Mohan S.

Shrikhande, who spoke on ‘Delsarte polynomial and designs’, and I, who spoke on ‘Designs on association schemes’.

In the evening of 24 October, the speakers in this session all gathered for dinner at the Best Western. Mohan Shrikhande surprised and delighted everyone by bringing his father to the dinner. I was among those who managed to spend some time sitting next to him and talking about joint interests. During this conversation, Shrikhande senior told me that the proudest moment of his entire career was proving the Bose–Shrikhande–Parker theorem.

PART II

Graph Basics

2
Definitions

2.1 What Is a Graph?

‘Graph theory’ is a relatively recent addition to the mathematical canon. Some date its origins to the Swiss mathematician Leonhard Euler’s (1708–83) work on the bridges of Königsberg (now Kaliningrad) in 1736. The city was at the junction of two rivers forming an island, and seven bridges connected the four regions of the city (Fig. 2.1). According to legend, the citizens asked Euler whether it was possible to take a walk, crossing each of the bridges just once and returning to the starting point. Euler showed that this was not possible.

(The picture shows the layout of the bridges in Euler’s time. Since then, destruction in war and new construction have changed the layout. See [75] for a contemporary account.)

The problem is unchanged if we draw it in an abstract way as shown later in Fig. 2.2. Each of the four regions is represented by a dot, and each of the seven bridges by a line connecting two dots.

Such a diagram is a representation of a *graph*. However, the first usage of the term ‘graph’ was due to J. J. Sylvester [100] in 1878.

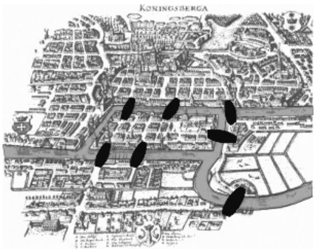

Figure 2.1 The bridges of Königsberg

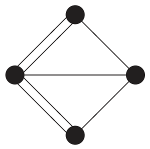

Figure 2.2 The bridges of Königsberg as a graph

Graphs play two main roles in mathematics and applications. First, as in Euler's example, they relate to *connectivity* properties of a structure. For the second role, suppose we have a network of radio transmitters, say for mobile phones. For reasons of basic physics, transmitters which are close together cannot both transmit on the same frequency. So the graph describing the network models *incompatibility*: if we think of frequencies as 'colours' assigned to the transmitters, then neighbouring transmitters must be given different colours. This application goes back to the famous *Four Colour Problem* from the nineteenth century, asking whether any map could be coloured with four colours so that countries sharing a border are given different colours.

We will refer several times to the history of graph theory. A good reference for this is the book by Biggs, Lloyd and Wilson [16].

2.2 Definitions and Basic Results

This section contains all the definitions and basic results that a beginning graph theory student might need. If you are familiar with these, skip to Section 2.3; you can refer back here when necessary. If you are learning graph theory, we have included exercises to illustrate the definitions.

Definition A *graph* $G = (V, E)$ consists of a non-empty collection V of points called its *vertices*, and a set E of unordered pairs of distinct vertices called its *edges*. The unordered pair of vertices $\{u, v\} \in E$ are called the *end vertices* of the edge $e = \{u, v\}$. In that case, the vertex u is said to be *adjacent* to the vertex v. Two edges e and e' are said to be *incident* if they have a common end vertex. $|V|$ is called the *order* of G, denoted by n or $n(G)$; and $|E|$ is called the *size* of G, denoted by m or $m(G)$. A graph G is *trivial* or *empty* if it has no edges.

We sometimes abbreviate the edge $\{v, w\}$ to vw.

Note that this definition means that edges have no preferred direction, that no vertex is joined to itself, and that at most one edge joins a given pair of vertices. Graph theorists say that our graphs are *undirected*, *loopless* and *simple* (no *multiple edges*). In some parts of graph theory, these conditions are relaxed.

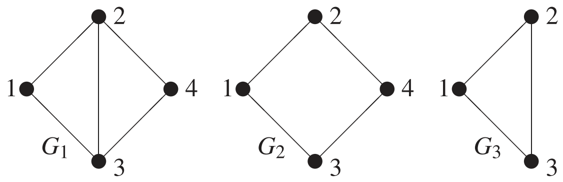

Figure 2.3 Spanning and induced subgraphs

(For example, the graph of the Königsberg bridges shown earlier is undirected and loopless but not simple.)

Exercise Show that there are eight graphs on a set of three vertices. How many 'essentially different' graphs are there? (By the term 'essentially different' we mean 'non-isomorphic graphs', which will be explained later.)

Definition A graph is *complete* if every pair of distinct vertices are the end vertices of an edge.

A complete graph on n vertices is denoted by K_n.

Exercise Prove that K_n has $\frac{n(n-1)}{2}$ edges.

Definition A graph $H = (V', E')$ is called a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. (This definition implies that, if $\{v, w\}$ is an edge in E', then both v and w belong to V'.) A subgraph H is a *spanning subgraph* if $V' = V$. H is called an *induced subgraph* if E' is the collection of all edges in G which has both its end vertices in V'. The induced subgraph with vertex set V' is denoted by $\langle V' \rangle$.

Figure 2.3 shows examples: G_2 is a spanning subgraph of G_1, while G_3 is an induced subgraph.

Definition A property P of a graph G is *vertex hereditary* if every induced subgraph of G has the property P. A graph H is a *forbidden subgraph* for a property P if any graph G that satisfies the property P cannot have H as an induced subgraph. A graph G is *H-free* if it does not have H as an induced subgraph.

Exercise Prove that, if a complete graph is a subgraph of a graph G, then it is an induced subgraph of G.

Definition The number of vertices adjacent to a vertex v is called the *degree* (or *valency*) of the vertex, denoted by $d(v)$. A vertex of degree 0 is called an *isolated vertex*, a vertex of degree one is called a *pendant vertex*, and a vertex of degree $n - 1$ is called a *universal vertex* (or *dominating vertex*).