1 Introduction

Philosophers have always been tantalised by the notion of INFINITY and the complicated puzzles that it raises in various philosophical contexts. The nature and characteristics of the infinite and how (if at all) it can be instantiated in the world have been the subject of long-standing philosophical discussions. Philosophers of different eras and traditions of thought have engaged with the infinite through various approaches and from different perspectives. But there is no doubt that some of the most exciting episodes of such engagements have occurred in the medieval traditions of Jewish, Christian, and Islamic philosophy. Philosophers from these traditions discussed a wide variety of issues regarding the notion of INFINITY and its instances in the world (if any). Medieval encounters with the notion of INFINITY have various aspects and can be approached from different angles. Medieval arguments for the impossibility of one or another sort of infinity form one such aspect. Some of the most significant ideas about infinity, which have played a crucial role in the evolution of our understanding of this notion, were introduced and/or developed in the context of the medieval arguments for finitism. In the wide spectrum of these arguments, those that are related, in one way or another, to the problem of the possibility of infinities of different sizes seem to have significant historical and philosophical connections to our modern concept of infinity. Nevertheless, many aspects of the historical development of such arguments and their philosophical significance are still unexplored. This Element aims to shed light on previously uninvestigated corners of medieval finitism by discussing two main groups of the most important medieval arguments that engage with the notion of INFINITIES OF DIFFERENT SIZES.[1] Given this specific scope, I refrain from engaging with medieval arguments for infinitism in general or for the existence of infinities of different sizes in particular.[2]

My focus in this study is primarily on the *mathematical* aspects of medieval finitism. However, it is important to note that extensive discussions of finitism can rarely (if at all) be found in medieval *mathematical* works. Medieval scholars usually investigated the infinite in either the works of *theology* and *metaphysics* (in connection to issues like the eternity of creation, arguments for the existence of God, the infinity of a chain of causally related elements, and the infinity of the objects of God's knowledge or power) or the works of *physics* (in connection to issues like the infinity of the world, the infinity and continuity of motion, the infinity of power, the atomistic structure of the material world,

[1] For two seminal studies focused on historical engagements with the idea of infinities of different sizes, see Davenport (1999) and Mancosu (2009).
[2] Such arguments are extensively discussed in Mancosu (forthcoming).

and the existence of vacuum). That is why the primary concern of many medieval arguments discussed in this Element is not mathematical. Nevertheless, we cannot reach a comprehensive picture of the historical evolution of the notion of MATHEMATICAL INFINITY without careful analyses of these arguments.

This Element is structured as follows. Section 2 illustrates the definition and some of the characteristics that medieval philosophers typically considered for infinity under the influence of the ancient Greek philosophers and, in particular, Aristotle (d. 322 BCE). In the same section, I also discuss some (though by no means all) significant distinctions regarding the various types of infinities that medieval philosophers employed to develop their theories of infinities. Without a precise understanding of those distinctions, we cannot easily detect subtle differences among diverse medieval approaches to finitism. Different versions of what I call 'the Equality Argument' are discussed in Section 3. This argument relies on the assumption that there cannot be infinities of different sizes. Although this assumption does not sound true from our contemporary perspective, it was accepted by many ancient and medieval philosophers. Section 4 provides a detailed analysis of another influential finitist argument, which is usually called 'the Mapping Argument'. The mature version of the Mapping Argument was presented by Ibn Sīnā (d. 1037) – who was referred to in the Latin tradition by 'Avicenna' – through the refinement of an earlier, less accurate version by al-Kindī (d. 870). The philosophical significance of the main ideas developed in the context of debates concerning the soundness of these arguments and their relevance to our contemporary conception of mathematical infinity will be discussed in Section 5, where this Element concludes.

Before closing this introduction, I must clarify that although this Element addresses all three medieval Jewish, Christian, and Islamic traditions of philosophy, my primary focus is on the Islamic tradition. This is not only because I am more familiar with this tradition but also because of two other things. First, in the secondary literature in Western languages, medieval Arabic-Islamic theories of infinity are studied no more than their Jewish counterparts and far less than the Christian ones. Second, and more importantly, the most significant discussions of the Equality and Mapping Arguments in Jewish and Christian philosophy are historically posterior to and, in many cases, inspired by earlier discussions of these arguments in the Islamic tradition. In each subsequent section, I analyse the views of medieval thinkers in historical order. As we will see, Muslim figures take precedence in many of these sections. Admittedly, many sophisticated discussions of infinity in the other two traditions have had no anticipation in the Islamic tradition. For example, many of the arguments discussed in the fourteenth-century Latin philosophy (usually considered the most important period of the medieval debates about infinity) have no counterparts in the Islamic tradition. However, I do not discuss those arguments in this Element because, as I have already mentioned, I am mainly concerned with the Equality and Mapping Arguments.

2 Definition and Characteristics of Infinity

Infinitude is limitlessness. However, limitlessness can be understood in two different ways. As Fakhr al-Dīn al-Rāzī (d. 1210), a Muslim theologian and philosopher, puts it:

T1. Al-Rāzī (1990, *Eastern Investigations*, vol. 1, p. 297)

> What is literally said [to be limitless] is said either in the way of simple negation (*al-salb*) or in the way of metathetic affirmation (*al-ʿudūl*). As for what is said in the way of simple negation, it [i.e., to say that it is limitless in the way of simple negation] is to take away from that thing the meaning because of which it is correct to describe that thing as having a limit. And that [meaning] is quantity. This is like what is said of God Most High that He is limitless and of the point that it is limitless. As for what is [said to be limitless] in the way of metathetic affirmation, there is something because of which it is [in principle] correct to describe that thing as having a realised limit, but no limit is [in fact] realised.[3]

According to this passage, limitlessness can be understood in two different senses. A thing can be limitless because it lacks quantity. Such a thing is not capable of having a limit. Thus, it would be a category mistake to talk about the limit of it. In the same sense that talking about the colour of justice is a category mistake, talking about the limit of God or of a point is a category mistake. The limitlessness of such things must be taken in the way of simple negation. Limit is by no means attributable to such things. By contrast, things that possess quantity can, in principle, have a limit. Now, if such a thing – for example, a line – has no limit, the limitlessness of it must be understood through metathetic affirmation. To better grasp the distinction made in the passage, consider the sentences ‘justice is colourless’ and ‘the glass is colourless’. The former sentence – assuming that it is true – expresses a simple negation because justice cannot have a colour. Colour is by no means attributable to justice. However, the latter sentence can be interpreted as expressing a metathetic affirmation because the glass has no colour while, in principle, it could have a colour.[4] In our discussion of infinity, we are concerned with things that are

[3] Unless otherwise mentioned, all the translations from Arabic and Persian are mine. Accordingly, when I cite a work that includes both an original text in Arabic or Persian and its English translation, the page numbers refer to the Arabic or Persian part of the cited work.

[4] The origin of the distinction between simple negation and metathetic affirmation is Aristotle’s *De Interpretatione* 10. To see how this distinction is usually understood in the context of the

limitless in the sense of metathetic affirmation. These are the things that are unlimited, though they could have been limited.

T1 alludes to the general point that, in the framework of the Aristotelian categories, infinity must be considered an attribute of quantities. Quantities are either discrete or continuous. Number and language are examples of discrete quantities; and line, surface, body, time, and place are examples of continuous quantities (*Categories* 6, 4b20–25).[5] Thus, roughly speaking, infinity must be considered an attribute of *magnitudes* and *multitudes*.[6] According to Aristotle, 'something is infinite if, taking it quantity by quantity, we can always take something outside' (*Physics* III.6, 207a7–8). This definition seems to be universally accepted by medieval philosophers. Some of them explicitly endorsed this definition – or some paraphrase of it – in their works. To give a couple of examples, Ibn Sīnā (2009, *The Physics of* The Healing, chapter III.7, § 3) contends that infinite things are those which 'whatever you take from them, you always find something outside of them'.[7] Instead of appealing to a repetitive process of taking from infinity, Ibn Sīnā defines infinity by

Aristotelian logic, consider a sentence '*a* is not F'. If this sentence is understood as expressing a simple negation, then it says that it is not the case that *a* is F. Thus, the sentence in question can be true regardless of whether *a* exists and whether it is capable of having F or not-F as a property. On the other hand, if that sentence is taken as expressing a metathetic affirmation, then it says that it is the case that *a* is not-F. Given the existential import of the affirmative claims, this sentence is true only if all the following conditions are satisfied: (a) *a* exists, (b) F and not-F are in principle attributable to *a* (or, equivalently, *a* is in principle capable of being F or not-F), and (c) as a matter of fact, a does have the property of not-F. Regarding the engagements of the philosophers of the classical period of Islamic philosophy with this distinction, see Thom (2008), Hodges (2012), and Kaukua (2020). The first paper addresses the account of al-Fārābī (d. 950), and the latter two focus on the view of Ibn Sīnā, which was the primary source for the majority of discussions concerning this distinction in the postclassical Islamic philosophy.

5 All the translations of Aristotle's terms and phrases are borrowed from Aristotle (1984, *The Complete Works of Aristotle*). In this specific translation, 'language' is taken to be the translation of the Greek term 'λόγος'. Other translators have selected 'speech' as the translation of this term. In any case, as it is explicitly mentioned in *Categories* 6, 4b32, what Aristotle here means by 'λόγος' is the spoken language, which is constituted of a series of sounds and can be 'measured by long and short syllables'. So, it is comprised of distinct units that can be counted. This might explain why language is considered a discrete quantity. Nevertheless, many scholars believe that it is not really clear why language must be included in discrete quantities. This unclarity is intensified by the fact that there is no reference to language in Aristotle's discussion of categories in *Metaphysics* V.13. 1020a7–32. On Aristotle's account of quantity, see, among others, Studtmann (2004).

6 Hereafter, for the sake of simplicity and unless otherwise specified, by a 'magnitude', I mean a straight line that represents a one-dimensional magnitude (e.g., weight or distance). Accordingly, by 'the length of a magnitude', I mean the length of the line that represents that magnitude. By setting a convention and taking a magnitude of a certain finite length as our measuring unit, we can represent numbers by magnitudes: number n can be represented by a magnitude of the length of n units. However, the possibility of making such conventions does not undermine the fact that magnitudes in themselves are continuous quantities. I will later clarify how a 'multitude' must be understood.

7 See also Ibn Sīnā (2009, *The Physics of* The Healing, chapter III.7, § 2 and chapter III.9, § 1). For Ibn Sīnā's definition of infinity, see McGinnis (2010, section 4) and Zarepour (2020, section 2).

a single take. He says that infinity is such that no matter how big what you take from it is, something remains. Although it is not explicitly stated, it must be assumed that what is taken is itself finite. The same definition, with a slightly different phrasing, is endorsed by Fakhr al-Dīn al-Rāzī (1990, *Eastern Investigations*, vol. 1, p. 297–98): an infinity is such that 'when you take whatever amount of it that you wish, you find something outside of it without the need for returning [what is taken]'. By adding the phrase 'without the need for returning', he probably means that you can take more and more from an infinity, and even if what is taken is never returned, there always remains something other than what has been taken so far.

Other medieval philosophers either did not provide any explicit definition of infinity or offered other definitions that are somehow compatible with the Aristotelian definition of infinity. We will soon visit some of such definitions. Nevertheless, to the best of my knowledge, there was no significant criticism of the Aristotelian definition of infinity in medieval philosophy. At least not when we are only concerned with physical and mathematical infinity without touching on other things, for example, the qualitative infinity of God. The Aristotelian definition seems to be compatible with all the medieval discussions of infinity that are investigated in the following sections.

Aristotle makes two crucial distinctions about infinity. One between infinity *by addition* and *by division*, and the other between *actual* and *potential* infinity (*Physics* III.6, 206a14–25). Roughly speaking, a totality is infinite by addition if and only if it is (or at least can be conceived as being) formed by the successive addition of parts each of which has a similar finite quantity (or, less technically, size). For example, a straight line AB that starts from A and extends infinitely in the direction of B is infinite by addition because it can be conceived as being formed by the successive addition of a segment of a finite length, such as d (Fig. 1a).[8] On the other hand, a totality is infinite by division if, with no limit, it can be successively divided into smaller parts. For example, a finite line CD can be halved infinitely many times by being successively divided at D_1, D_2, D_3, . . . so that, for every $n \geq 1$, $CD_n = 2CD_{n+1}$ (Fig. 1b). CD is infinite by division but not by addition. To explain the idea of infinity by division using the aforementioned Aristotelian definition of infinity, it can be said that

[8] To be accurate, this form of referring to an infinite line is misleading and incompatible with the standards of modern mathematics. This is because it leaves the impression that 'B' – in the same manner as 'A' – refers to a point. However, this should not be the case because otherwise 'AB' refers to a finite line segment that is bounded with A and B. Nevertheless, this is how infinite lines are referred to in many medieval texts. See, for example, T14. Thus, I remain faithful to their reference style, hoping that the contexts of the following discussions of infinite lines will spare the readers from potential misunderstanding caused by this style. In visualisation, the bounded side of a line is represented by a bullet point and the infinitely extending side of it by an arrow point.

CAMBRIDGE
Cambridge University Press & Assessment
978-1-009-61843-4 — Medieval Finitism
Mohammad Saleh Zarepour
Excerpt
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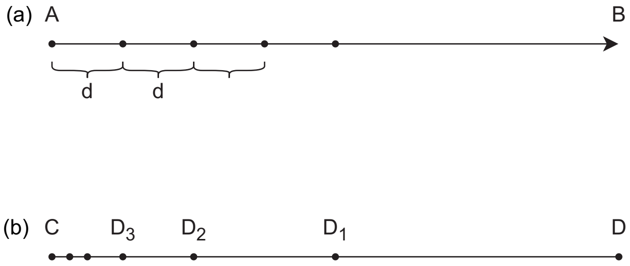

Figure 1 Infinity by addition and infinity by division.

a magnitude is infinite by division if and only if, no matter how many times you divide it into smaller parts, it is always possible to make a further division. The distinction between infinity by addition and by division provides a conceptual tool to differentiate the talk of infinitely big things from that of infinitely small things. This explains the natural association of the notion of INFINITY BY DIVISION with that of CONTINUITY. But what we are concerned with in this Element is mainly infinity by addition. More precisely, the primary aim of the two types of finitist arguments discussed in this study is to reject the possibility of certain sorts of infinity by addition.[9]

In broad terms, if the process of addition or division by which an infinity is formed is already completed and all the parts or components of that infinity coexist simultaneously, then that infinity is *actual*; otherwise, it is *potential*. If a magnitude is being extended infinitely by successively adding segments of a certain length while its current length is finite, then that line is only potentially infinite. Similarly, if a finite magnitude is, in principle, divisible into infinitely many parts but is not yet so divided, its infinity (i.e., the infinitude of the multitude of all its division) must be considered potential, or so Aristotle suggested.[10] As we will see in the following sections, the distinction between actual and potential infinity plays a crucial role in the medieval accounts of infinity. However, it is important to note that not all medieval philosophers share similar interpretations of the notions of ACTUALITY and POTENTIALITY. As a result of various modifications that medieval philosophers proposed to these notions, there are examples of

[9] However, it must be noted that if something is infinite by division, the number of the divisions that can be made in that thing is infinite by addition. So, the notions of INFINITY BY DIVISION and INFINITY BY ADDITION, though distinct, are related to each other. The relation between these two conceptions of infinity is clearly visible in passages like T2 and T3.

[10] Aristotle's conception of infinity is studied, among others, by Hintikka (1966), Lear (1980), Kouremenos (1995), Bowin (2007), Coope (2012), Nawar (2015), and Cooper (2016).

infinite totalities that are taken as actual infinities by some philosophers and as potential infinities by others.[11]

Along with these distinctions about various types of infinity, we should take note of some of the most important characteristics that medieval philosophers considered for infinity. In particular, there are two ideas about infinity that many medieval views regarding the size of infinity are developed either based on or in reaction to them. The first idea goes as follows:

Equality of Infinities (EI): All comparable infinities are equal to each other. No infinity is greater or lesser than another.

In the next section, we will see that the long history of the discussions of infinity in which this idea or something in its vicinity is presupposed goes back at least to Lucretius (d. circa 55 BCE). **EI** was accepted as an incontrovertible axiom by many medieval philosophers. One might think that, for example, the infinite benevolence of God is not comparable to an infinite line considered in geometry. They are not of the same species. Nor can they be compared to each other quantitatively. Thus, it does not make sense to ask which one is greater, or so one might contend. However, different infinite geometrical lines are of the same species and comparable to each other. Therefore, if **EI** is true, we must conclude that no infinite line can be greater or lesser than the others. They are all equal to each other. To put it more cautiously, they are all of the same size. In general, in the context of the forthcoming discussions, when it is said that two things are equal, it merely means that those things are equal in terms of quantity.

A rationale behind **EI** could be that if something is infinite, it must be limitless. Moreover, if something is limitless, it must, in a sense, encompass everything. So, nothing can be greater than an infinity. Not even another infinity can surpass it. As it is stated by John Philoponus (d. 570) in his *Against Aristotle on the Eternity of the World* (2014, fr. 132, p. 144), it is 'impossible that ⟨anything⟩ should be greater than the infinite, or that the infinite should be increased'.[12] Also, **EI** might

[11] An important medieval distinction that I do not touch on in this Element is the distinction between the *categorematic* and *syncategorematic* senses of infinity, which is closely related to the distinction between *actual* and *potential* infinities. On the origin of the distinction between categorematic and syncategorematic infinities and its role in the medieval Latin discussions of the theories of infinity and continuity, see, among others, Geach (1967), Kretzmann (1982), Murdoch (1982, pp. 567–68), Duhem (1985, chapter 1), Uckelman (2015), and Moore (2019, section 3.3).

[12] The angle brackets are by the translator. The original text of *Against Aristotle* is lost. Nevertheless, a large part of this treatise is now reconstructed based on the fragments quoted in Greek, Arabic, and, in one case, Syriac sources. The most reliable fragments are those quoted by Simplicius (d. 560), who had access to the original treatise, in his commentaries on Aristotle's *Physics* and *On the Heavens*. Fortunately, Simplicius's quotes form the largest portion of the reconstructed treatise.