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1 First- and Second-Order ODE

1.1 Introduction

The first-order equations–linear and non-linear–and the second-order linear equations, with constant or variable coefficients, are considered in this chapter. For solving the first-order equations, familiar methods such as the method of separation of variables or a method that can be reduced to this are used. Also discussed are the exact differential equations or those equations that can be reduced to this form using a suitable *integrating factor* (*IF*). We also emphasize the peculiarities that may arise in an initial value problem (IVP) when sufficient conditions imposed in the Cauchy–Peano existence theorem or in the method of Picard's iterations are not satisfied. Many exercises deal with the maximal interval of existence of a solution to an IVP.

Only second-order linear equations are considered here. The non-linear equations or, more generally, the two-dimensional systems of first-order equations are treated in Chapter 5 on qualitative analysis. The treatment of equations with constant coefficients is straightforward. The equations with variable coefficients are more difficult to deal with, and, in general, it is not possible to obtain the solution in explicit form. However, the structure of solutions to the homogeneous and inhomogeneous equations is well-understood.

A general first-order ordinary differential equation (ODE) takes the form $f(t, x(t), x'(t)) = 0$, where f is a given function and $x = x(t)$ is the unknown function to be determined. A general theory for the above equation is rather difficult. A more realistic equation for which a general theory can be developed is given by the regular form $x'(t) = f(t, x(t))$, and the corresponding IVP is

$$x'(t) = f(t, x(t)),\ t \in I,\ x(t_0) = x_0. \tag{1.1.1}$$

It should be noted that the initial time t_0 and the initial value x_0 come from the domain of the definition of f. If $\mathbf{f} = (f_1, \cdots, f_n)$ is a vector-valued (n-dimensional) and $\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^n$ is a vector, then equation (1.1.1) is written in boldface as $\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}(t)), t \in I, \mathbf{x}(t_0) = \mathbf{x}_0$ and is a system of n equations in n unknowns $\mathbf{x}(t) = (x_1(t), \cdots x_n(t))$. It is readily seen that any nth-order equation $x^{(n)}(t) = f(t, x(t), x'(t), \cdots x^{(n-1)}(t))$ can be reduced to the first-order system by substituting $x_1(t) = x(t), x_2(t) = x_1'(t), \cdots, x_{n-1}(t) = x_{n-2}'(t)$ and $x_n(t) = x_{n-1}'(t) = f(t, x_1(t), \cdots, x_n(t))$, that is, $\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t))$ with

$$\mathbf{F}(t, \mathbf{x}(t)) = (x_2(t), \cdots, x_n(t), f(t, x_1(t), \cdots, x_n(t)).$$

First-Order Linear Equations: A general regular form of the first-order equation is given by

$$x'(t) + p(t)x(t) = q(t), \tag{1.1.2}$$

where we assume that p and q are given continuous functions defined on an interval I. This equation can be solved using an IF $\mu(t) = \exp\left(\int^t p(\tau)\, d\tau\right)$. Multiplying equation (1.1.2) by μ, the equation reduces to an integrable (exact) form as

$$(\mu x)' = \mu q,$$

and the solution is given by $x(t) = (\mu(t))^{-1}\left(\int^t \mu q\, d\tau + C\right)$.

The ODE $x'(t) = f(t, x(t))$ is said to be an *exact equation* if it can be reduced to an integrable (exact) form that can be directly integrated to get the solution. That is, the equation can be written as $\dfrac{d}{dt}\phi(t, x(t)) = 0$ for some two-variable function ϕ. If we write the first-order equation in a general form as $M(t, x) + N(t, x)x'(t) = 0$, then it is exact if and only if $\dfrac{\partial M}{\partial x} = \dfrac{\partial N}{\partial t}$, assuming the functions M and N are C^1 in a domain D in the (t, x) plane. If it is not exact, we look for a function μ so that the equation $\mu M + \mu N x' = 0$ is exact. Such a μ, when exists, is called an IF.

Thus, we have a reasonable complete theory for solving the first order linear equations. Such a complete procedure is not available for the second- (and higher-) order linear equations of the form

$$x''(t) + p(t)x'(t) + q(t)x(t) = r(t),\ t \in I. \tag{1.1.3}$$

Nevertheless, we have the following result regarding the existence theory and structure of the solutions.

Theorem 1.1. Assume p, q and r are continuous functions in a compact interval I. Then, the following statements hold:

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(i) For arbitrary $x_0, x_1 \in \mathbb{R}$, equation (1.1.3) has a unique solution x in some sub-interval $I(t_0)$ of I containing t_0 satisfying the initial conditions $x(t_0) = x_0$, $x'(t_0) = x_1$.

(ii) Let S be the set of all solutions of the homogeneous equation (1.1.3) (i.e., with $r \equiv 0$). Then, S is a vector space and $\dim(S) = 2$. In other words, the homogeneous equation has two linearly independent solutions.

(iii) Let $\tilde{S}$ be the set of all solutions of equation (1.1.3). Then, $\tilde{S} = S + x_p$, where x_p is any particular solution of equation (1.1.3). ■

Thus, $\tilde{S}$ is a hyperplane. In general, there is no specific procedure to determine two linearly independent solutions of the homogeneous equation. However, if one (non-trivial) solution $x_1(t)$ is known, then the second linearly independent solution can be obtained by the method of order reduction (also called the method of variation of parameters or constants). More precisely, we look for a second solution of the form $x_2(t) = c(t)x_1(t)$, where the function $c(t)$ needs to be determined. It is readily seen that c satisfies the equation $x_1(t)c''(t) + (2x_1'(t) + p(t)x_1(t))c'(t) = 0$, as x_1 is a solution. Thus, c can be obtained by performing two integrations.

On the other hand, if the homogeneous equation has constant coefficients, then the solutions can be explicitly written down. More precisely, consider the equation $ax''(t) + bx'(t) + cx(t) = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Then, the general solution is given by

$$x(t) = \begin{cases} Ae^{r_1 t} + Be^{r_2 t} \text{ if } b^2 - 4ac > 0, \\ (A + Bt)e^{\alpha t} \text{ if } b^2 - 4ac = 0, \\ e^{\alpha t}(A\cos\beta t + B\sin\beta t) \text{ if } b^2 - 4ac < 0. \end{cases}$$

Here, A and B are arbitrary constants, $\alpha = -b/(2a)$, $\beta = \sqrt{4ac - b^2}/(2a)$ and

$$r_1 = (-b + \sqrt{b^2 - 4ac})/(2a), \quad r_2 = (-b - \sqrt{b^2 - 4ac})/(2a).$$

Returning to the IVP (1.1.1), we have general results regarding the existence, uniqueness and continuous dependence of solutions. For existence, the continuity of f is a sufficient condition (Peano's theorem), whereas the uniqueness requires a stronger assumption of Lipschitz continuity of f. We state these results in the sequel. First, we recall Gronwall's inequality, a powerful and interesting result in establishing uniqueness result.

Theorem 1.2 (Gronwall's Inequality)**.** Suppose that p and q are continuous real-valued functions defined on $[a, b]$ with $q(t) \geqslant 0$ on $[a, b]$. Assume p and q satisfy the inequality

$$p(t) \leqslant C + k\int_{t_0}^{t} q(s)p(s)ds,$$

for all $t \in [a, b]$ and $t \geqslant t_0$, where $t_0 \in [a, b]$ is fixed and C and k are constants with $k \geqslant 0$. Then,

$$p(t) \leqslant C \exp\left(k \int_{t_0}^{t} q(s)\, ds\right),$$

for all $t \in [a, b]$, $t \geqslant t_0$; for $t \leqslant t_0$, interchange the limits in the above integrals. ■

It is easy to verify that a differentiable function $x = x(t)$ defined on $I = (a, b)$ is a solution of IVP (1.1.1) if and only if x is a solution of the corresponding integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau,\ t \in (a, b). \tag{1.1.4}$$

This is termed as the integral formulation. However, we remark that to define a solution of equation (1.1.4), we do not require the differentiability of x. Thus, it is possible to define the solution in a weaker sense that a continuous function is a solution if it satisfies the integral formulation. This weaker notion of a solution even allows to relax the condition of continuity on the function f. This has many applications including control theory, where the function f need not be a continuous function when discontinuous controls are employed. Nevertheless, it is an interesting fact that if we start with a continuous function f and continuous solution x, then we can establish that x is differentiable and it is a classical solution, that is, a solution of equation (1.1.1).

By the above observation, it suffices to prove the existence and uniqueness of a continuous solution to the integral equation, when f is a continuous function.

Definition 1.3 (Lipschitz Continuity)**.** A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be locally Lipschitz continuous in D if, for any $x_0 \in D$, there exists neighbourhood N_{x_0} of x_0 and an $\alpha = \alpha(x_0) > 0$ such that

$$|f(x) - f(y)| \leqslant \alpha |x - y|, \text{ for all } x, y \in N_{x_0}.$$

The function $f : D \subset \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz continuous (or globally Lipschitz continuous) in D if there is a constant $\alpha > 0$ such that $|f(x) - f(y)| \leqslant \alpha\, |x - y|$, for all $x, y \in D$. ■

The smallest such constant α is known as the Lipschitz constant of f. It is easy to see that if f is differentiable and $\beta = \sup\limits_{x \in D} |f'(x)| < \infty$, then f is Lipschitz in D with Lipschitz constant $\alpha \leqslant \beta$. We now extend this definition to a vector-valued function. Let $D \subset \mathbb{R}^n$ be a domain.

Definition 1.4 (Lipschitz Continuity). A function $\mathbf{f}(t, \mathbf{x}) : (a, b) \times D \to \mathbb{R}^n$ is said to be Lipschitz continuous (globally in D) with respect to $\mathbf{x}$ if there exists $\alpha > 0$ such that $|\mathbf{f}(t, \mathbf{x_1}) - \mathbf{f}(t, \mathbf{x_2})| \leqslant \alpha|\mathbf{x_1} - \mathbf{x_2}|$, for all $(t, \mathbf{x_1})$ and $(t, \mathbf{x_2})$ in $(a, b) \times D$. ■

The smallest such constant α is known as the Lipschitz constant of $\mathbf{f}$. We can also define local Lipschitzness analogously.

Theorem 1.5 (Uniqueness). Let a and b be positive real numbers. Suppose that $f = f(t, x)$ is continuous in the rectangle $\mathcal{R} = \{(t, x) \in \mathbb{R}^2 : |t - t_0| \leqslant a, |x - x_0| \leqslant b\}$ and Lipschitz continuous with respect to x in $\mathcal{R}$. Then, the IVP (1.1.1) has at most one solution. ■

Regarding existence, we have the following couple of theorems.

Theorem 1.6 (Picard's Existence Theorem). Let D be an open connected set in $\mathbb{R}^2$. Assume $f : D \to \mathbb{R}$ satisfies the following conditions:

(i) f is continuous in D.

(ii) f is Lipschitz continuous with respect to x in D with Lipschitz constant $\alpha > 0$.

Let $(t_0, y_0) \in D$ and a and b are positive constants such that the rectangle $\mathcal{R}$ is a subset of D and put $M = \max\limits_{(t,y)\in\mathcal{R}} |f(t, y)|$ and $h = \min(a, b/M)$. Then, the IVP (1.1.1) has a unique solution in the interval $|t - t_0| \leqslant h$. ■

The proof is based on the convergence of the sequence of iterates known as *Picard's iterates*, which are recursively defined by

$$x_k(t) = x_0 + \int_{t_0}^{t} f(\tau, x_{k-1}(\tau))d\tau, \ t \in (a, b), \tag{1.1.5}$$

for $k = 2, 3, \ldots$, and the first iterate is $x_1(t) = x_0$, the initial value. Under the assumptions stated in the theorem, we can show that the sequence $\{x_k\}$ converges uniformly to some x in $C(I)$, the space of continuous functions defined on I and equipped with sup norm. It is then shown that this x is the unique solution of the integral equation.

The assumption of Lipschitz continuity is needed only to prove the uniqueness. However, we do have the existence result with only continuity assumption on f. Indeed, the proof of Picard's theorem uses Lipschitz continuity even for existence.

Theorem 1.7 (Cauchy—Peano Existence Theorem). Let a, b and $\mathcal{R}$ be as in Theorem 1.6. Assume $f(t, y)$ is continuous on the rectangle $\mathcal{R}$. Then, there exists a solution to the IVP (1.1.1) in the interval $|t - t_0| \leqslant h$, where $h = \min\left(a, \dfrac{b}{M}\right)$ and $M = \max\limits_{\mathcal{R}} |f|$. ■

We remark that the interval of existence given by the previous theorems need not be the best possible and the solution may exist on larger intervals leading to the concept of the *maximal interval of existence*. This is nothing but the union of all intervals of existence of the IVP, which is an open interval. We now move on to the discussion of the continuous dependence of the solution to the IVP on the data: the initial condition and the function f. This concept is crucial in applications; it essentially says that if the error in input data (initial value x_0, dynamics f) is small in appropriate norms, then the error in solution is also small. We consider a more general IVP with an input function u in the dynamics as:

$$x'(t) = f(t, x(t), u(t)),\ t \in I,\ x(t_0) = x_0. \tag{1.1.6}$$

Such a problem has wide applications in many areas. For example, in control theory, such input functions are known as *controls* that can be manipulated to obtain a desired trajectory to reach a desired state. For a given function u, the standard theory will give the existence and uniqueness with appropriate assumptions on f, and there will be a unique solution x starting from x_0. However, a control problem is that whether an input function (control) u can be chosen so that the trajectory x reaches a pre-designed state x_1 at a prescribed time T. In general, this is not true, and we may see examples in Chapter 2 on the linear systems. We now state the continuous dependence result.

Theorem 1.8 (Continuous Dependence)**.** Let $\mathcal{R}$ be as in Theorem 1.5. Suppose $f,\ \tilde{f} \in C(\mathcal{R})$ are Lipschitz continuous with respect to x variable, with Lipschitz constants $\alpha,\ \tilde{\alpha}$ respectively. Let x and $\tilde{x}$ be, respectively, the solutions of the IVP $x' = f(t,x),\ x(t_0) = x_0$ and $\tilde{x}' = \tilde{f}(t,\tilde{x}),\ \tilde{x}(\tilde{t}_0) = \tilde{x}_0$ in some closed intervals $I_1,\ I_2$ containing t_0 and $\tilde{t}_0$. For small $|t_0 - \tilde{t}_0|$, let I be any finite interval containing t_0 and $\tilde{t}_0$, where both x and $\tilde{x}$ are defined. Then,

$$\max_{t\in I} |x(t) - \tilde{x}(t)| \leqslant \left(|x_0 - \tilde{x}_0| + |I| \max_{\mathcal{R}} |f(t,x) - \tilde{f}(t,x)| + M|t_0 - \tilde{t}_0| \right) e^{\alpha_0 |I|},$$

where $|I|$ is the length of the interval I, $M = \max\left(\max_{\mathcal{R}} |f|,\ \max_{\mathcal{R}} |\tilde{f}| \right)$ and $\alpha_0 = \min(\alpha, \tilde{\alpha})$. ■

1.2 Some Applications

We now discuss applications of the first-order equations to some two-dimensional geometric problems. As is customary, we use (x,y) to denote a point in the plane, and x is considered an independent variable and y a dependent variable.

1.2.1 The Envelope of a Family of Curves

Consider a one-parameter family of smooth two-dimensional curves represented by

$$F(x, y, \alpha) = 0. \tag{1.2.1}$$

Here, α denotes the parameter which is a real number; each α gives rise to a curve in the family. By smoothness, we mean that the partial derivatives $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$ and $\dfrac{\partial F}{\partial \alpha}$ exist and are continuous. For example, for the family of straight lines passing through the origin, the slope of the line plays the role of the parameter; for the family of circles with same radius and different centres, all lying on a straight line, the parameter comes through the centres of the circles. On the other hand, the family of circles with distinct radii and distinct centres, all lying on a straight line, is an example of more than one-parameter family of curves.

Definition 1.9. A curve L is said to be an *envelope* of a one-parameter family of curves if at each point the curve L touches a curve of the family and different curves in the family touch the curve L at different points. That is, at each point on L, the curve L is tangential to exactly one curve of the family and vice versa. ■

Given a family of curves as in equation (1.2.1), we will now determine an envelope of it using the following procedure. Suppose the curve given by $y = \varphi(x)$, where φ is a C^1 function, is an envelope. Since at each point (x, y), the envelope touches a curve in the family, which in turn determines a value of the parameter α, obviously depending on the point (x, y). Denote this by $\alpha = \alpha(x, y)$. Thus, $F(x, y, \alpha(x, y)) = 0$ for each point on the envelope. Assuming that $\alpha(x, y)$ is a differentiable function, we obtain after differentiation with respect to x the following relation:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial \alpha}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y} y'\right) = 0, \tag{1.2.2}$$

at each point of the envelope. The slope of the tangent to the curve of the family (1.2.1) at the point (x, y) is obtained from the equation

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' = 0, \tag{1.2.3}$$

as α is a constant as far as the family of curves is concerned. Assume $\dfrac{\partial F}{\partial y} \neq 0$; we reverse the roles of x and y if $\dfrac{\partial F}{\partial y} = 0$ but $\dfrac{\partial F}{\partial x} \neq 0$. Since the envelope touches every curve of the family, by definition, the tangent of the envelope is the same as that of the tangent of the curve of the family, where the envelope touches the curve. It therefore follows from equations (1.2.2) and (1.2.3) that $\frac{\partial F}{\partial \alpha}\left(\frac{\partial \alpha}{\partial x}+\frac{\partial \alpha}{\partial y}y'\right)=0$. On the envelope, $\frac{\partial \alpha}{\partial x}+\frac{\partial \alpha}{\partial y}y' \neq 0$ as α is not a constant on the envelope. Thus, on the envelope, the following two relations must hold:

$$F(x,y,\alpha)=0 \quad \text{and} \quad \frac{\partial F}{\partial \alpha}(x,y,\alpha)=0. \tag{1.2.4}$$

Conversely, if, by eliminating α from the two equations in (1.2.4), we can obtain a function $y=\varphi(x)$, where φ is a C^1 function and α is not a constant on this curve, then $y=\varphi(x)$ determines an envelope of the family in equation (1.2.1).

The points (x,y) where both $\frac{\partial F}{\partial x}=0$ and $\frac{\partial F}{\partial y}=0$ vanish are termed as *singular points*. It is not difficult to show that the singular points also satisfy equation (1.2.4). Thus, equation (1.2.4) either determines an envelope or the locus of singular points or a combination of both.

Numerical Range of a Matrix: We now take a short digression into linear algebra [17], [21]. If A is a real or complex $n\times n$ matrix, its *numerical range*, denoted by $W(A)$, is the set of complex numbers $\{(Az,z) : \|z\|=1\}$, where $(\cdot,\cdot)$ and $\|\cdot\|$ denote the standard scalar product and the norm in $\mathbb{C}^n$, respectively. It is not difficult to show that the spectrum (the set of eigenvalues) of A is a subset of $W(A)$. Clearly, $W(A)$ is a bounded subset as $|(Az,z)| \leqslant \|A\|$ for all $\|z\|=1$. It is also a convex set. Furthermore, $W(U^*AU)=W(A)$ for any unitary matrix U. We use this to determine $W(A)$ when $n=2$. By a well-known result, there is a unitary matrix $\mathbf{U}$ such that $U^*AU=\begin{bmatrix}\lambda_1 & m\\ 0 & \lambda_2\end{bmatrix}$, where λ_1 and λ_2 are the eigenvalues of A and $m=0$ if and only if A is a normal matrix. Therefore, $W(A)=\{\lambda_1|z_1|^2+\lambda_2|z_2|^2+m\bar{z}_1z_2\}$, where z_1 and z_2 are arbitrary complex numbers such that $|z_1|^2+|z_2|^2=1$. Put $w=\lambda_1|z_1|^2+\lambda_2|z_2|^2+m\bar{z}_1z_2$ and $t=|z_1|^2$. Then, $w=t\lambda_1+(1-t)\lambda_2+|m|\sqrt{t(1-t)}\,e^{i\gamma}$, where t varies in $[0,1]$ and γ is an arbitrary real number. From this, we conclude that $W(A)$ is the line segment joining λ_1 and λ_2, if $m=0$; if $\lambda_1=\lambda_2=\lambda$, say, then $W(A)$ is the circular disk centred at λ and radius $|m|/2$. Now, assume that λ_1 and λ_2 are distinct and $m\neq 0$. Observe that $\frac{w-\lambda_2}{\lambda_1-\lambda_2}=t+\frac{|m|\sqrt{t(1-t)}}{\lambda_1-\lambda_2}e^{i\gamma}$. Thus, if we put $z=\frac{w-\lambda_2}{\lambda_1-\lambda_2}=x+iy$, where x and y are real, it follows $(x-t)^2+y^2=a^2t(1-t)$, where $a=\frac{|m|}{|\lambda_1-\lambda_2|}$. This is a family of circles with $t\in[0.1]$ playing the role of a parameter. Therefore, if we find the envelope of this family of circles, that in turn will determine the required numerical range of the 2×2 matrix A. This will be done in an exercise.

1.2.2 Orthogonal and Isogonal Trajectories

Consider a one-parameter family of smooth curves in the plane:

$$F(x, y, \alpha) = 0. \tag{1.2.5}$$

Definition 1.10. The curve that intersects every curve of the family in equation (1.2.5) at a *constant* angle is called an *isogonal trajectory*. If the constant angle is a right angle, it is called an *orthogonal trajectory*. ■

An isogonal trajectory is also called an *oblique trajectory*. If two straight lines with slopes m_1 and m_2 intersect at a right angle, then $m_1 m_2 = -1$; an appropriate statement may be made while considering the lines parallel to axes. Using this simple fact, it is now straightforward to find the orthogonal trajectories.

The slope of the tangent line to any curve in the family (1.2.5) at a point (x, y) is determined by the equation

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' = 0.$$

Eliminating α from equation (1.2.5), we obtain the differential equation

$$G\left(x, y, \frac{dy}{dx}\right) = 0. \tag{1.2.6}$$

Therefore, at (x, y), the differential equation satisfied by the orthogonal trajectories is

$$G\left[x, y, -\left(\frac{dy}{dx}\right)^{-1}\right] = 0. \tag{1.2.7}$$

Suppose $\tilde{F}(x, y, \beta) = 0$ describes the general solution of equation (1.2.7) with an arbitrary parameter β. This yields a family of orthogonal trajectories of the given family of curves (1.2.5).

A family of isogonal trajectories to the family of curves in equation (1.2.5) can be found using the following elementary observation. If two straight lines with slopes m_1 and m_2 intersect at an (acute) angle γ, then $\tan \gamma = \dfrac{|m_2 - m_1|}{|m_1 m_2 + 1|}$ or $m_1 = \dfrac{m_2 - \kappa}{\kappa m_2 + 1}$ with $\kappa = \tan \gamma$. The differential equation satisfied by the family of curves (1.2.5) is the same as equation (1.2.6), namely

$$G\left(x, y, \frac{dy}{dx}\right) = 0.$$

Therefore, the differential equation satisfied by an isogonal trajectory is given by

$$G\left[x, y, \frac{p - \kappa}{1 + \kappa p}\right] = 0, \tag{1.2.8}$$

where $p = \dfrac{dy}{dx}$ and κ is as above. Solving this equation for a general solution with one arbitrary constant gives a family of isogonal trajectories.

The general references are [46], [10], [11] and [29].

1.3 Exercises

Exercise 1.1

Discuss the existence and uniqueness of the following IVP:

(1) $x_1' = x_2,\ x_2' = -\dfrac{g}{l}\ \sin(x_1) - \dfrac{k}{m}\ x_2,\ x_1(0) = 0,\ x_2(0) = 0.$

(2) $x_1' = x_2 - x_1(x_1^2 + x_2^2),\ x_2' = -x_1 - x_2(x_1^2 + x_2^2),\ x_1(0) = x_2(0) = 0.$

Exercise 1.2

Discuss the solvability of the IVP $y' = -\text{sgn}(y)$, $y(0) = y_0 > 0$. Here,

$$\text{sgn}(y) = \begin{cases} 1 \text{ if } y \geqslant 0, \\ -1 \text{ if } y < 0. \end{cases}$$

Exercise 1.3

State the conditions under which the following differential equations will have a unique solution:

(i) The nth-order non-linear equation $y^{(n)}(t) = g(t, y(t), y^{(1)}(t), \dots, y^{(n-1)}(t))$.

(ii) The nth-order linear non-homogeneous equation

$$y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y^{(1)}(t) + a_n(t)y(t) = b(t).$$

Exercise 1.4

In the following exercises, describe the domain of definition of the given function and corresponding initial condition, find the maximal interval (α, β) of existence of the solution and find its limits as t approaches α, β:

$$(1) \quad y' = \frac{1}{1 + y^2}, \qquad (2) \quad y' = \frac{1}{ty}, \qquad (3) \quad y' = \frac{1}{2y}.$$

Exercise 1.5

Assume the IVP $y' = ay(t) - by^2(t)$, $y(t_0) = y_0$, where a and b are positive real numbers and t_0, $y_0 \in \mathbb{R}$, has a unique solution $y = y(t)$ in an interval (t_1, t_2) containing t_0.

(1) Without attempting an explicit representation of the solution, show that y satisfies the relation $\text{sgn}(y(t)(y(t) - (a/b))) = \text{sgn}(y_0(y_0 - (a/b)))$.