Preface

There are, to the best of our knowledge, four forces at play in the universe. At the very largest scales – those of planets or stars or galaxies – the force of gravity dominates. At the very smallest distances, the two nuclear forces hold sway. For everything in between, it is force of electromagnetism that rules.

At the atomic scale, electromagnetism (admittedly in conjunction with some basic quantum effects) governs the interactions between atoms and molecules. It is the force that underlies the periodic table of elements, giving rise to all of chemistry and, through this, much of biology. It is the force which binds atoms together into solids and liquids. And it is the force which is responsible for the incredible range of properties that different materials exhibit.

At the macroscopic scale, electromagnetism manifests itself in the familiar phenomena that give the force its name. In the case of electricity, this means everything from rubbing a balloon on your head and sticking it on the wall, through to the fact that you can plug any appliance into the wall and be pretty confident that it will work. For magnetism, this means everything from the shopping list stuck to your fridge door, through to trains in Japan which levitate above the rail. Harnessing these powers through the invention of the electric dynamo and motor has transformed the planet and our lives on it.

As if this wasn't enough, there is much more to the force of electromagnetism for it is, quite literally, responsible for everything you've ever seen. It is the force that gives rise to light itself.

Rather remarkably, a full description of the force of electromagnetism is contained in four simple and elegant equations. These are known as the *Maxwell equations*. There are few places in physics, or indeed in any other subject, where such a richly diverse set of phenomena flows from so little. The purpose of this book is to introduce the Maxwell equations and to extract some of the many stories they contain.

There is also a second, broader theme that runs through this book, and that is the notion of a field. On our journey through theoretical physics, from Newton's laws of motion to a modern view of the universe, there are many twists and turns. But there are two great, paradigm-changing leaps. One is quantum mechanics. The other is field theory. And this book is our first step towards understanding fields.

Electromagnetism is not, it turns out, the simplest field theory. It is, in fact, rather subtle, with the subtleties arising largely because of a strange property known as *gauge symmetry*. As this book progresses, we will get to grips with this idea of gauge symmetry. This is important, not only for electromagnetism, but also for many subsequent developments in theoretical physics, from the Standard Model to general relativity.

Finally, it is worth stressing that the Maxwell equations are the first time that we meet what we would call a "fundamental law of physics". The equations remain largely unchanged as we look more closely at the universe, undergoing just a few small embellishments (and, admittedly, one big embellishment when we later view these same equations through the lens of quantum mechanics). Crucially, the Maxwell equations provide a blueprint that is followed by all other fundamental forces. This, ultimately, is the reason why we should study them closely and learn as much as we can.

How to Read This Book

This book starts by presenting the Maxwell equations in Chapter 1 and then proceeds to pick them apart, piece by piece, over the subsequent three chapters, to find the magic hiding inside. We first look at situations involving only electric fields in Chapter 2, then situations involving only magnetic fields in Chapter 3. Then we look at the novel features that arise when both electric and magnetic fields move together in Chapter 4. Prominent among these novel features is the phenomenon called electromagnetic waves, known more colloquially as "light".

It makes sense to read these first four chapters in order, as each builds on the previous one. This material is really the essence of electromagnetism. It is the material that makes up the bulk of our second year undergraduate course in Cambridge.

Chapter 5 describes the relativistic formulation of electromagnetism. This includes a reminder of the basics of special relativity, but you'll be better equipped if you're already comfortable with the subject from the first volume on Classical Mechanics. Our emphasis in this chapter shifts somewhat, from explaining what electromagnetism can do, to looking more closely at the underlying structure of the equations. Indeed, it's in this section that we start to really appreciate the theoretical beauty and simplicity of the Maxwell equations.

This theme continues in Chapter 6. Here we describe the Lagrangian and Hamiltonian approaches to Maxwell theory and use this to highlight some interesting features and applications. Again, we recapitulate the basics, but you will be better served if you're already happy with these topics, both of which are covered in the first volume on Classical Mechanics. Among the applications that these new techniques lend themselves to are topological insulators and the physics of superconductors and the Higgs boson. The topics covered in this chapter are less standard fare in electromagnetic textbooks. However, the kind of methods that we will meet will become increasingly important as we proceed to the later volumes in this series.

Chapters 7 and 8 both explore aspects of electromagnetism in more detail and can be read in any order. This is material that we cover in our third year undergraduate course in Cambridge. Chapter 7 is concerned with electromagnetic waves. We'll describe how these waves are generated by accelerating charged particles, and look a little at how light scatters. This chapter uses the relativistic notation that was introduced in Chapter 5 but can be read independently of Chapter 6.

Chapter 8 describes what happens to electric and magnetic fields inside materials. This is where we meet the fields $\mathbf{D}$ and $\mathbf{H}$, as opposed to $\mathbf{E}$ and $\mathbf{B}$. The first few sections are reasonably elementary and rely only on Chapters 1 through 4. However, the content picks up pace as the chapter progresses. By the time we get to Section 8.7, describing the screening of charge in a material, we throw caution to the wind. This material is more typically found in (reasonably advanced) textbooks on solid state physics, and we will be drawing on ideas from the subsequent volumes on Quantum Mechanics and Statistical Physics. If you are still on the early stages of your journey through theoretical physics, I would advise holding off a little before hitting this section!

The final part of this book is formed of a series of mathematical appendices, providing a self-contained review of results in vector calculus. Everything in these appendices is assumed background knowledge for the bulk of this book. Indeed, some of it is assumed for the previous volume on Classical

Mechanics, so the chronology is not entirely consistent here. Nonetheless, the theory of electromagnetism is the first time that ideas in physics are so firmly entwined with vector calculus, and I think it's useful to have a stand alone resource that explains the basic ideas and proofs. In addition, this is the precursor to differential geometry, which will be covered in some detail in the later volume on General Relativity.

Just One Book Among Many

This is second volume in a series of $N \gg 1$ books.

There are many paths through theoretical physics and, correspondingly, many different routes through this series of books. I've tried to order things in a way that makes logical sense, but it is by no means a perfect nor a unique ordering. For that reason, you should treat these books more like a "choose your own adventure" series, picking your way through the chapters that you need or that excite you, rather than reading each book in turn.

The Maxwell equations are a cornerstone of theoretical physics, but it will actually be some time before they will be needed for later developments. There will be plenty of discussion of electric and magnetic fields in the volumes on Quantum Mechanics, Statistical Mechanics, and Condensed Matter but, for the most part, these fields will be viewed as something fixed and our interest will be in how charged particles respond to them. For this, we don't need the full Maxwell equations, just the Lorentz force law

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \ .$$

There are a number of small, but important, exceptions to this state of affairs, notably when we discuss photons in the later volumes, but they are few and far between.

This means that you should certainly feel free to learn quantum mechanics before you learn the Maxwell equations and, if you're so minded, it would be possible to push on and study statistical mechanics and condensed matter without a good Maxwell grounding although, the further you went, the more perverse this would become.

At some point, however, the Maxwell equations become indispensable. This is partly because electromagnetism is our first example of a field theory, and this means that some of the techniques that we learn will be useful for other, seemingly unrelated, classical field theories, including in the volume on Fluid Mechanics and in the later volume on General Relativity. But mostly the importance of the Maxwell equations lies in the fact that they are a fundamental law of nature and, as such, play an increasingly prominent role as we progress along our journey to learn physics. This will be especially true in the later volumes on Quantum Field Theory and the Standard Model.

Problems

I have not included any exercises in this series of books. However, the Faculty of Mathematics at the University of Cambridge has long had a policy of making all problems (and, indeed, exam questions) publicly available. Problem sheets aligned with the material in this book can be downloaded from:

www.damtp.cam.ac.uk/user/tong/books/electro.html

An errata for the book can also be found on this webpage.

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The appendices of this book were written for a course on vector calculus that I taught to first year undergraduates at Cambridge. I have helped myself to a number of examples from the excellent lecture notes of my colleagues who taught this course before me, in particular Jonathan Evans and Ben Allanach. I'm particularly grateful to Jonathan for useful discussions on how best to teach this material. I've also benefitted from the detailed notes of Stephen Cowley. My thanks to Julia Gog and Maria Gutierrez for suggesting the application of the divergence theorem to predator–prey models

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CAMBRIDGE

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David Tong
Excerpt
More Information

1 The Basics

Still those papers lay before me —
Problems made express to bore me,
When a silent change came o'er me,
In my hard uneasy chair.
Fire and fog, and candle faded,
Spectral forms the room invaded,
Little creatures, that paraded
On the problems lying there.

James Clerk Maxwell's ode to his undergraduate examples sheets.

The beauty of electromagnetism is that we get so much from so little. The entirety of the theory is contained within a handful of short equations that govern the dynamics of charges and fields. For this reason, we start this book with a short chapter that tells you everything there is to know about electromagnetism. The rest of the book consists only of commentary.

1.1 Charge and Current

Every particle in the universe is endowed with a number of properties that determine how it responds to each of the four fundamental forces. For the force of gravity, this property is mass. For the force of electromagnetism, the property is called *electric charge*.

Electric charge is a number. Importantly, charge can be positive or negative. It can also be zero, in which case the particle is unaffected by the force of electromagnetism.

The SI unit of charge is the *coulomb*, denoted by C. It is, like all SI units, a parochial measure, convenient for human activity rather than informed by the underlying laws of physics. (We'll learn more about how the coulomb is defined in Section 3.5). At a fundamental level, nature provides us with a much better unit of charge. This follows from the fact that charge is quantised: every free particle has an electric charge that is an integer multiple of the charge of an electron. By (a slightly annoying) convention, the electron is taken to have negative electric charge $-e$, with

$$e = 1.602176634 \times 10^{-19}\ \mathrm{C}\ . \tag{1.1}$$

A much more natural unit would be to simply count charge as $q = ne$ with $n \in \mathbb{Z}$. Electrons then have charge -1 while protons have charge $+1$ and neutrons have charge 0. While this is natural, in this book we will bow to convention and stick with SI units.

As an aside, I sneaked in the weasley word "free" above in an attempt to mitigate the fact that the sentence is tending towards a lie. This is because the charge of quarks is actually fractional, with $q = -e/3$ and $q = 2e/3$. But quarks are trapped inside protons and neutrons, a prison from which they find it hard to escape. More pertinently, the fractional charge of quarks doesn't change the spirit of the discussion: charge is still quantised, but we just need to change the basic unit.

The theory of electromagnetism involves the dynamics of two, conceptually very different, objects. These are particles and fields. Ultimately, as we progress on our journey through physics, we will learn that everything is fields. But, for this book, we are stuck with both. And they are not, it turns out, always the most comfortable of bedfellows, with a point-like particle causing a singularity in the continuous fields. Sometimes this is straightforward to deal with, sometimes less so.

One cheap, but convenient way to deal with this is to replace point-like particles with their *charge density* $\rho(\mathbf{x}, t)$. This is the charge per unit volume. The total charge Q in a given region V is then simply the integral

$$Q = \int_V d^3x\ \rho(\mathbf{x}, t)\ . \tag{1.2}$$

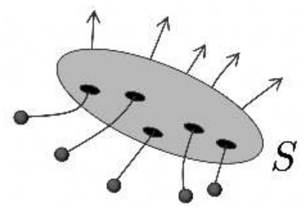
In most situations, we will consider smooth charge densities, which can be thought of as arising from averaging over many point-like particles. But working with charge densities also leaves open the option of returning to a single point-like particle of charge q, moving on some trajectory $\mathbf{r}(t)$, by writing $\rho(\mathbf{x}, t) = q\delta^3(\mathbf{x} - \mathbf{r}(t))$ where $\delta^3(\mathbf{x} - \mathbf{r}(t))$ is a singular function known as the *delta function*. (We will define this function in Section 2.2.) This function has the property that all the charge sits at a mathematical point. This is where the singularities raise their head when we try to marry particles with fields and we will have to understand how to deal with this as we go along.

We will also need to describe the movement of charge from one place to another. This is captured by a quantity known as the *current density* $\mathbf{J}(\mathbf{x}, t)$, defined as follows: for every surface S, the integral

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{1.3}$$

counts the charge per unit time passing through S. (Here $d\mathbf{S}$ is the unit normal to S. These mathematical issues are described in more detail in the appendices.) The quantity I is called the *current*. The equation (1.3) is telling us that current density is the current per unit area.

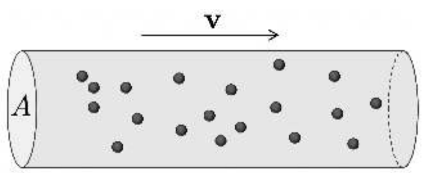
Equation (1.3) is a rather indirect definition of the current density. To get a more intuitive picture, consider a continuous charge distribution in which the velocity of a small volume, at point $\mathbf{x}$, is given by $\mathbf{v}(\mathbf{x}, t)$. Then the current density is



$$\mathbf{J} = \rho \mathbf{v} \ . \tag{1.4}$$

In particular, if a single particle is moving with velocity $\mathbf{v} = \dot{\mathbf{r}}(t)$, the current density will be $\mathbf{J} = q\mathbf{v}\delta^3(\mathbf{x} - \mathbf{r}(t))$. This is illustrated in the figure, where the underlying charged particles are shown as balls, moving through the surface S.

As a simple example, consider electrons moving through a wire. We model the wire as a long cylinder of cross-sectional area A, as shown in the figure. The electrons move with velocity $\mathbf{v}$, parallel to the axis of the wire. (In reality, the electrons will have some distribution of speeds; we take $\mathbf{v}$ to be their average velocity.) If there are n electrons per unit volume, each with charge q, then the charge density is $\rho = nq$ and the current density is $\mathbf{J} = nq\mathbf{v}$. The current itself is $I = |\mathbf{J}|A$.



Throughout this book, the current density $\mathbf{J}$ plays a much more prominent role than the current I. For this reason, we will often refer to $\mathbf{J}$ simply as the "current", although we'll be more careful with the terminology when there is any possibility for confusion.

1.1.1 The Conservation Law

The single most important property of electric charge is that it's conserved. This, of course, means that the total charge in a system can't change.

We met many conservation laws in the first book on Classical Mechanics. These included conservation of energy, momentum, and angular momentum. In all cases, the conserved quantity was a number and that number didn't change as the system evolved.

Now, however, we've made the leap to field theories. And that's an important leap because it means that we're forced to think about where the conserved quantity sits in space, rather than just a total number. In other words, we need to think about charge density $\rho(\mathbf{x}, t)$, and not just the integrated charge $Q(t)$. And that brings something new to the table, because things that are conserved in our universe are always conserved *locally*.

For example, an electric charge sitting in the palm of your hand can't just disappear and turn up on Jupiter. That would certainly satisfy a "global" conservation of charge, but that's not the way the universe works. If the electric charge disappears from your hand, then most likely it has fallen off and is now sitting on the floor. Or, said more precisely, it must have moved to a neighbouring region of space.

The property of local conservation means that the charge density $\rho(\mathbf{x}, t)$ can change in time only if some charge moves to a nearby region. Which, in turn, means that there must be a compensating current flowing from one point of space to another. This property is captured by the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \ . \tag{1.5}$$

This is an important equation. It is the equation that tells us something is conserved locally.

To see why the continuity equation captures the right physics, consider the change in the total charge Q contained in some fixed region V. From (1.2), the change in the charge is given by

$$\frac{dQ}{dt} = \int_V d^3x \ \frac{\partial \rho}{\partial t} = -\int_V d^3x \ \nabla \cdot \mathbf{J} = -\int_S \mathbf{J} \cdot d\mathbf{S} \tag{1.6}$$

where, in the final equality, we've used the divergence theorem so the area S is the boundary of the region V, which we write as $S = \partial V$. From (1.3), we