Games of No Chance 6
MSRI Publications
Volume **71**, 2025

Combinatorial game theory monoids and their absolute restrictions: a survey

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The classic order relation in Combinatorial Game Theory asserts that, given a winning convention, a game is greater than or equal to another whenever Left can exchange the second for the first in any disjunctive sum, and she can do this regardless of the other component, without incurring any disadvantage. In the early developments of Combinatorial Game Theory, the "other component" encompassed any possible game form, and a rich normal play theory was built by Berlekamp, Conway, and Guy (1976–1982), via a celebrated *local* comparison procedure. It turns out that the normal play convention is a lucky case, and recent research on other conventions therefore often restricts the ranges of games to various subclasses. In the case of misère play, it is possible to obtain partially ordered monoids with more structure by imposing restrictions. The same is true in scoring play. Furthermore, Absolute Combinatorial Game Theory was recently developed as a unifying tool for a local game comparison that generalizes the normal play findings, provided that the restricting set is *parental* (among a few other closure properties), meaning that any pair of finite, nonempty subsets of games from the restriction is permissible as sets of options for another game in the set. This survey aims to provide a concise overview of the current advancements in the study of these structures.

1. Background and purpose of the overview

Combinatorial games are two-player games with perfect information (no hidden information as in some card games) and no chance moves (no dice), where the players move alternately. When the current player has no more moves, the game

Santos is the corresponding author. His work is funded by national funds through the Fundação para a Ciência e a Tecnologia (FCT), I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).
MSC2020: 91A46, 06F05.
Keywords: combinatorial game theory, local comparison, atomic reversibility, conjugate property.

ends and some given convention determines the result. Combinatorial Game Theory (CGT) is the branch of mathematics that studies combinatorial games.

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In the case of misère play, it is possible to obtain partially ordered monoids with more structure by imposing restrictions. The same is true in scoring play. Furthermore, Absolute Combinatorial Game Theory [Larsson et al. 2025b] was recently developed as a unifying tool for a local game comparison that generalizes the normal play findings, provided that the restriction is *parental* (among a few other closure properties), meaning that any pair of finite, nonempty subsets of games from the restriction is permissible as sets of options for another game in the set.[1] This survey aims to provide a concise overview of current advancements in the study of these structures.

We will be interested in short games and we assume familiarity with basic CGT concepts such as winning conventions, game forms, options, followers, outcomes, disjunctive sum, game inequality, game equivalence, game reductions, canonical forms, and so on. All of these concepts are presented and discussed in the classic references [Albert et al. 2007; Berlekamp et al. 1982a; 1982b; Conway 1976; Siegel 2013]. We intend for this to be a reasonably advanced survey. If you are a reader unfamiliar with CGT, acquiring knowledge of the fundamental concepts in the specialized literature will be necessary.

Under the normal play convention, it is well known that $G \succcurlyeq 0$ if and only if $G \in \mathscr{L} \cup \mathscr{P}$, and we will refer to this result as the *Fundamental Theorem of Normal Play* (FTNP). The ultimate reason for this theorem is that if Left has a winning strategy in a game X, then she also has it in $G + X$. She can use a "local response strategy", meaning that, in $G + X$, Left responds to Right's moves in

[1] There are almost as many naming conventions in this thrilling new study as there are coauthors; the two emphasized terms in this sentence have various other suggestive namings as follows: the term "local" is synonymous with *constructive*, *algorithmic*, *recursive*, *computable*, and *play* in various works within the literature, and the term "parental" is synonymous with *dicotic-closed* and *absolute*. In the near future, we hope for some agreement on various concepts. For now, we are satisfied to explain their contexts.

a component with a move in that same component, as if she were playing it in isolation. Under the normal play convention, Left will make the last move in both components, and, consequently, in the disjunctive sum as a whole. The FTNP is sufficient to prove that there is a perfect matching between the order relation and the outcome classes. That is: $G \succ 0$ if and only if $G \in \mathscr{L}$; $G = 0$ if and only if $G \in \mathscr{P}$; $G \parallel 0$ if and only if $G \in \mathscr{N}$; and $G \prec 0$ if and only if $G \in \mathscr{R}$. It is also well known that all game forms have inverses, meaning that game forms, together with the disjunctive sum, form a *group structure*.

By writing $-G = \{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\}$ to denote the conjugate of G, i.e., the game in which players' roles are reversed, it is straightforward to verify that $G - G \in \mathscr{P}$, which means that $G - G = 0$. This time, instead of the local response strategy, the argument employs an opposite strategy commonly referred to as the *Tweedledee–Tweedledum strategy*. Whenever a player makes a move in one component, the opponent plays the symmetrical move in the other component. This use of symmetry ensures the last move for the player who plays second. Even when considering other conventions, for the sake of simplicity in writing, we will continue to denote $\{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\}$ as $-G$, even though it may not be the inverse of G in other contexts. With the notion of conjugate in mind, the FTNP is also sufficient to provide an easy *local* way to compare G with H. To determine if $G \succcurlyeq H$, one simply needs to check if Left wins the game $G - H$ playing second. All of these concepts are specific to normal play, and this is important for what will follow. Local response strategies, Tweedledee–Tweedledum strategies, the matching between the order relation and the outcomes, and the group structure are all things that are lost in other conventions. Following the previous observations, regarding the normal play convention, the following list of facts has long been known and was first detailed in [Berlekamp et al. 1982a; 1982b; Conway 1976]. Some terminology, and the subdivision of item (2) into four parts is ours. We wish to highlight the idea of this subdivision, because it generalizes normal play to other settings.

(1) There is a *local* comparison procedure, which involves evaluating whether Left wins the game $G - H$ playing second. In other words, $G \succcurlyeq H$ if and only if
   (i) for each $G^R \in G^{\mathcal{R}}$, there is a $G^{RL} \in G^{L\mathcal{R}}$ such that $G^{RL} - H \succcurlyeq 0$ or there is an $H^R \in H^{\mathcal{R}}$ such that $G^R - H^R \succcurlyeq 0$, and
   (ii) for each $H^L \in H^{\mathcal{L}}$, there is a $G^L \in G^{\mathcal{L}}$ such that $G^L - H^L \succcurlyeq 0$ or there is an $H^{LR} \in H^{L\mathcal{R}}$ such that $G - H^{LR} \succcurlyeq 0$.

(2) There are four types of reductions:
   (i) Domination: If G is a game with two Left options $G^{L_1}, G^{L_2} \in G^{\mathcal{L}}$ and $G^{L_2} \succcurlyeq G^{L_1}$, then $G = \{G^{\mathcal{L}} \setminus \{G^{L_1}\} \mid G^{\mathcal{R}}\}$.

(ii) Nonatomic Reversibility: If G is a game with a Left option $G^L \in G^{\mathcal{L}}$ and there is a G^{LR} such that $G \succcurlyeq G^{LR}$, then, if $G^{LR\mathcal{L}}$ is nonempty, $G = \{G^{\mathcal{L}} \setminus \{G^L\}, G^{LR\mathcal{L}} \mid G^{\mathcal{R}}\}$.

(iii) Atomic Reversibility: If G is a game with a Left option $G^L \in G^{\mathcal{L}}$ and there is a G^{LR} such that $G \succcurlyeq G^{LR}$, then, if $G^{LR\mathcal{L}}$ is empty, $G = \{G^{\mathcal{L}} \setminus \{G^L\} \mid G^{\mathcal{R}}\}$.

(iv) Replacement by Zero: If $G = \{G^L \mid G^R\}$ is a game with a single move for each player, and both G^L and G^R are atomic reversible options, then $G = 0$.

Consider any game form under normal play. By exhaustively applying these reductions, in any order, the end product is unique, and is referred to as the game's *canonical form*.

(3) All games are invertible, and the inverse of a game G is its conjugate $-G$.

In normal play, the three latter reductions in item (2) merge into one, since the statement of nonatomic reversibility encompasses both atomic reversibility and replacement by zero. So, in that case, only the two reductions of domination and reversibility are mentioned in the literature.

In misère play, some items from the list fail or may require some modification. Nevertheless, when considering different classes of distinguishing games, it is possible to find monoids with some very interesting properties. In addition to the full universe of games ($\mathcal{M}$), the universes of dead-ending ($\mathcal{E}$) and dicotic ($\mathcal{D}$) games are notable examples.

Also, in scoring play, some items from the list fail. In this case, besides the full universe of games (or the Stewart universe, $\boldsymbol{S}$), notable examples include the universe of guaranteed scoring games ($\boldsymbol{Gs}$) and the Ettinger universe ($\boldsymbol{E}$). We will detail later the fact that the replacement by zero is always valid in classical nonscoring restrictions but may fail under the scoring play convention.

In this document, while adapting to literature, these universes are denoted by $\mathcal{M}$, $\mathcal{E}$, $\mathcal{D}$, $\boldsymbol{S}$, $\boldsymbol{Gs}$, and $\boldsymbol{E}$, respectively. It is important to emphasize that if, for example, a section pertains to $\mathcal{D}$, the symbol $\succcurlyeq$ will refer to the inequality defined in $\mathcal{D}$, avoiding the need to write $\succcurlyeq_{\mathcal{D}}$. This type of restriction is commonly referred to in the literature as *modulo* $\mathcal{D}$. In other words, if a section is about a specific universe $\mathcal{U}$, everything mentioned in that section will be modulo $\mathcal{U}$.

The concept of a *universe* of games is fundamental. A universe is a set (possibly a restriction) of games under a given convention that satisfies standard closure properties.

The idea of studying these restrictions originates from what we informally call *the waiting problem*. Consider misère play, and suppose that Left has no options in a game G, i.e., G is a *Left-end*. When playing first in misère play, Left wins G when played in isolation. But when played together with another game H, written as $G + H$, then there is the possibility that Left has to play to some $G + H^L$. Now, Right, who was *waiting* for the opportunity, can play in G, say to $G^R + H^L$. That move can lead to Left's defeat, with the moves that Left has to make in a follower of G being the decisive factor in her loss. The game $G = \{\varnothing \mid \mathbf{3}\}$, in which Left has no moves, but where Right can give Left three consecutive moves, is an example of a game that may cause this problem. In general, occurrences of the waiting problem reduce the richness of mathematical structure, resulting in fewer smaller equivalence classes of games born by a given birthday, and so on. Fortunately, some restrictions prevent the occurrence of this problem and, as a bonus, invite many recreational-play rulesets.

A game is a *dicot* if, in each subposition, either both players can move, or neither player can move. It is easy to observe that games such as $\{\varnothing \mid \mathbf{3}\}$ are not dicots, and consequently the waiting problem does not occur in a dicotic universe/restriction. A game is dead-ending if, whenever a player has no available move at a subposition, they have no move in any follower of that subposition. Again, it is easy to observe that games such as $\{\varnothing \mid \mathbf{3}\}$ are not dead-ending, and hence the waiting problem does not occur in a dead-ending universe/restriction. Of course, every dicot is also dead-ending, but the converse is not true. These classes eliminate the waiting problem, as the defining properties ensure that a player cannot play again in a component after running out of moves in it.

Regarding scoring play, the Ettinger universe is dicotic. In this universe, when a player runs out of moves in a component, the resulting score is some real number, and both players are left with no moves in that component. On the other hand, the universe of guaranteed scoring is analogous to a dead-ending universe from nonscoring theory. When a player runs out of moves in a component, if that component were played in isolation, a final score $s \in \mathbb{R}$ would be obtained. The guaranteed property assures that, with other components in play, the score of s in that component cannot become worse with respect to that player, even if play continues there. The waiting problem is again avoided.

Each of these restrictions has its algebraic structure, with larger equivalence classes than the corresponding full universe, and their analyses have been conducted over the years, as we will detail further below. Related to this type of research, an important event was the development of Absolute Combinatorial Game Theory [Larsson et al. 2025b]: a unifying additive theory for standard restrictions in CGT. The main result of this is so crucial for this survey that we will begin to detail it now, starting with some general definitions. Absolute theory encompasses all so-called *parental* (also called *dicotic-closed*) universes: any game form constructed with a pair of nonempty finite sets of elements from the universe, as Left and Right options respectively, is also an element of the universe.

Definition 1 (Maintenance). Let G and H be games in a universe $\mathcal{U}$. The pair (G, H) satisfies the maintenance, $\operatorname{Maint}(G, H)$, if

$$\forall G^R(\exists G^{RL} \text{ such that } G^{RL} \succcurlyeq_{\mathcal{U}} H \text{ or } \exists H^R \text{ such that } G^R \succcurlyeq_{\mathcal{U}} H^R)$$

and

$$\forall H^L(\exists H^{LR} \text{ such that } G \succcurlyeq_{\mathcal{U}} H^{LR} \text{ or } \exists G^L \text{ such that } G^L \succcurlyeq_{\mathcal{U}} H^L).$$

It is helpful to decompose the standard outcomes, to take an explicit note of the winner depending on who starts, and we write $\mathrm{o} = (\mathrm{o}_L, \mathrm{o}_R)$, with $\mathrm{o}_L, \mathrm{o}_R \in \{\mathrm{L}, \mathrm{R}\}$. so that for example, $\mathscr{L} = (\mathrm{L}, \mathrm{L})$, $\mathscr{N} = (\mathrm{L}, \mathrm{R})$, and so on. Here, the convention is the total order $\mathrm{L} > \mathrm{R}$. In absolute theory, we usually call Left-ends instead Left-atomic games (roughly, atoms can be adorned with a “score”). Here, we use the two terms interchangeably.

Definition 2 (Proviso). Let G and H be games in a universe $\mathcal{U}$. The pair (G, H) satisfies the proviso $\operatorname{Proviso}(G, H)$ if the following two items hold:

(i) if H is Left-atomic, then, for any Left-atomic X, $\mathrm{o}_L(G+X) \geqslant \mathrm{o}_L(H+X)$;

(ii) if G is Right-atomic, then, for any Right-atomic X, $\mathrm{o}_R(G+X) \geqslant \mathrm{o}_R(H+X)$.

Theorem 3 (Absolute Comparison). *Let $\mathcal{U}$ be a parental universe. Then $G \succcurlyeq_{\mathcal{U}} H$ if and only if* $\operatorname{Maint}(G, H)$ *and* $\operatorname{Proviso}(G, H)$.

The absolute comparison gives rise to four observations. Firstly, note that, in normal play, if $H^{\mathcal{L}}$ is empty, then the first item of the proviso is trivially satisfied, since $\mathrm{o}_L(H+X) = \mathrm{R}$. On the other hand, if $G^{\mathcal{R}}$ is empty, then the second item of the proviso is trivially satisfied, since $\mathrm{o}_R(G+X) = \mathrm{L}$. Consequently, in normal play, if $\operatorname{Maint}(G, H)$, then $G \geqslant H$. In other words, the exception never occurs, and the proviso is unnecessary.

The second observation is that the exception of the proviso usually occurs in restrictions under the misère and scoring play conventions. For example, in $\mathcal{D}$, $\operatorname{Maint}(\{0 \mid *\}, 0)$ holds but $\operatorname{Proviso}(\{0 \mid *\}, 0)$ does not. Therefore, the issue arises that the absolute proviso involves all possible Left-ends and all possible Right-ends, which are obviously infinite in number. Fortunately, recent research has revealed that it is often not necessary to test all Left-ends and Right-ends, but only a few relevant ones. We have a *local* comparison procedure whenever it is possible to consider only a finite number of ends. Such a procedure is of utmost importance both in practice and theory. Therefore, an “absolutely” crucial question is the following.

Question 1. Given a parental universe, how/when can the proviso be reduced to a finite number of tests?

The third observation once again involves Left-ends and Right-ends; somehow, handling the empty set of options is one of the most delicate tasks in CGT. The

Cambridge University Press & Assessment
978-1-009-56552-3 — Games of No Chance 6
Edited by Urban Larsson
Excerpt
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main reason why the proviso is necessary when the convention is not normal play is due to the fact that the absence of options can be advantageous for a player. Having no moves can be beneficial under the misère play convention, because it may signify victory, and having no moves in scoring play can imply obtaining a high score. Thus, issues related to *atomic reversibility* arise. Let us start by analyzing the reversibility condition. If $G \succcurlyeq G^{LR}$, then in a disjunctive sum $G+X$, Left never plays to G^L+X with the intention of responding to $G^{LR}+X^L$ from a Right move to some $G^{LR}+X$. This is because it is no worse to move directly to $G+X^L$, as the condition implies $G+X^L \succcurlyeq G^{LR}+X^L$. This means that Left only chooses G^L+X if she intends to respond locally, in a follower of G, in case Right responds to $G^{LR}+X$. This is what is almost always referred to in the specialized literature and is indeed the concept that underlies reversibility.

Yet there is another idea that is less frequently mentioned, but is equally important. It arises from the following question: "When should Left play to G^L+X, if G^L is an atomic reversible option, that is, if $G^{LR\mathcal{L}}=\varnothing$, for some $G^{LR} \preccurlyeq G$?" If $X^{\mathcal{L}}$ is not empty, then there is an $X^L \in X^{\mathcal{L}}$ such that $o_R(G^L+X) \preccurlyeq o_R(G+X^L)$. This is because

$$\begin{aligned} o_R(G^L+X) &\preccurlyeq o_L(G^{LR}+X) && \text{(arbitrary choice)}\\ &= o_R(G^{LR}+X^L) && \text{(best choice, } G^{LR} \text{ is a Left-end)}\\ &\preccurlyeq o_R(G+X^L) && \text{(reversibility condition).}\end{aligned}$$

In other words, in game practice Left rarely needs to opt for an atomic reversible option. She only chooses such an option in a disjunctive sum if all the other components are Left-ends. However, in these cases, the atomic reversible option may be the only winning move, and as a result, except in a few cases where it is a sole option, it cannot reverse out. Nevertheless, it may be possible to replace it with a simpler atomic reversible option that allows for obtaining a useful "canonical form". In summary, an interesting *choice* for the replacement may be made or, at the very least, a *method* of making that choice can be indicated. Hence, a second crucial question arises.

Question 2. Given a parental universe, how and when can we solve atomic reversibility?

The fourth observation concerns invertible game forms. For some restrictions, the approach was to start by proving the Conjugate Property, establishing that, if G is invertible, then its inverse is $-G$. With that, in some cases, a simple characterization of the invertible elements of the structure was also proved. Thus, a third crucial question is the following.

Question 3. Is it true that the elements of a given parental universe satisfy the *Conjugate Property*, meaning that the inverse of each invertible element is its conjugate? Is there a simple way to characterize the invertible elements of the universe?

In normal play there is a unique canonical form (after exhaustive reductions in any order). It turns out that this is a fairly unique situation and, in general, one does not a priori get a unique form after reductions. For further discussion on this topic, we recommend [Larsson et al. 2016], [Larsson et al. 2025a], and in particular [Siegel 2025]. See also Section 5.2.

The upcoming overview follows a straightforward logic. It will elucidate the findings and responses that have evolved throughout the course of research to tackle the three first mentioned questions, henceforth denoted as Q1, Q2, and Q3. A concise summary of this overview is provided in Table 1.

Before we proceed, it is important for the reader to be aware of certain nomenclature issues. Often, when embarking on mathematical research in a new subject, different names may arise for the same concepts. The subject covered in this survey is no exception. For "local comparison", at least four other terms have been used: "subordinate comparison", "recursive comparison", "play comparison" and "constructive comparison". For the terms "end", "Left-end" and "Right-end", the terms "atomic", "Left-atomic", and "Right-atomic" have also been used. "Nonatomic reversibility" and "atomic reversibility" have been referred to as "open reversibility" and "end-reversibility". "Maintenance" has also been mentioned as "common normal part". Instead of "parental universe", the term "dicotic closed universe" has been used. In fact, a universe was originally defined as a class of game forms satisfying option closure, disjunctive sum closure, and conjugate closure, and containing the terminal positions. Recently, it has been proposed that the word "universe" be used only for parental universes. In this survey, we have made agnostic choices for each of these concepts. It is natural that, as the theory develops, these choices will become more stabilized in the specialized literature.

2. The misère play convention

The *classical conventions* are normal play and misère play. As mentioned, normal play does not require any special treatment in terms of restrictions. This section concerns popular restrictions of misère play.

2.1. ***Full misère, $\mathcal{M}$.*** For a long time, partizan games in misère play were considered essentially intractable. Then, in 2007, Mesdal and Ottaway [2007] showed the following highly relevant theorem concerning the full universe $\mathcal{M}$ of misère play. Necessarily, this must be the starting point of this section.

Theorem 4. *If G and H are misère games such that $H^{\mathcal{L}} = \varnothing$ and $G^{\mathcal{L}} \neq \varnothing$, then $G \not\succcurlyeq H$.*

In brief, the theorem by Mesdal and Ottaway indicates that G can only be greater than or equal to a Left-end H if it is also a Left-end itself. Of course, by symmetry, adopting Right's perspective, we also have that if $G^{\mathcal{R}} = \varnothing$ and $H^{\mathcal{R}} \neq \varnothing$, then $G \not\succcurlyeq H$. This result implies the following corollary, which states that every nonterminal game form is distinct from zero.

Corollary 5. *Let G be a misère game. If $G \not\cong \{\varnothing \mid \varnothing\}$, then $G \neq 0$.*

Corollary 5 immediately answers Q3: the only invertible game form is the terminal $0 = \{\varnothing \mid \varnothing\}$. That is, $\mathcal{M}$ is a monoid with no invertible elements other than the identity (otherwise known as a *reduced* monoid). Naturally, $\mathcal{M}$ satisfies the Conjugate Property, but in a trivial and unenlightening way.

Another interesting consequence of Theorem 4 pertains to the proviso. The proviso is a fundamental concept used in Theorem 3, but this consequence was not mentioned, as absolute theory had not yet been developed. Consider a pair (G, H) where $H^{\mathcal{L}} = \varnothing \implies G^{\mathcal{L}} = \varnothing$ and $G^{\mathcal{R}} = \varnothing \implies H^{\mathcal{R}} = \varnothing$, meaning that H cannot be a Left-end without G being one, and G cannot be a Right-end without H being one. It is relatively easy to prove that, under the misère play convention, a pair meeting these conditions and satisfying the maintenance also satisfies the proviso. Therefore, the answer to Q1 also follows from Theorem 4; if (G, H) does not belong to this family of pairs, then inevitably $G \not\succcurlyeq H$. The proviso in $\mathcal{M}$ is as restrictive as it can be.

Proviso of $\mathcal{M}$:

(i) if $H^{\mathcal{L}} = \varnothing$ then $G^{\mathcal{L}} = \varnothing$;

(ii) if $G^{\mathcal{R}} = \varnothing$ then $H^{\mathcal{R}} = \varnothing$.

Siegel [2015] then determined the full mathematical structure of $\mathcal{M}$. We observe once again that Theorem 4 led to an understanding of the reductions in $\mathcal{M}$ and, consequently, to the establishment of canonical forms. The reason for this lies in the fact that it is easy to verify that atomic reversibility and replacement by zero are reductions that *never occur*, thus answering question Q2. The rationale behind this is that it would imply the reversibility criterion $G \succcurlyeq G^{LR}$ where $G^{LR\mathcal{L}}$ is empty, which is something that the aforementioned theorem indicates cannot happen (G^{LR} is a Left-end, and G is not). Therefore, just as in the case of the normal play convention, reversibility can be uniquely stated through the statement of nonatomic reversibility. In summary, in both normal play and misère play, reversibility has the same statement. However, in the former convention, *there can be options reversing out*, while in the latter, that case *never occurs*. The fact that the reductions are the same in both conventions, although for different reasons, may have led to initial misunderstandings and a delay in the mathematical development of other restrictions that require four reductions instead of two.

2.2. *Dicotic misère, $\mathcal{D}$.* The best way to start this section is by addressing Q1. This is because there are no dicots that are Left-ends or Right-ends, except for the terminal form $0 = \{\varnothing \mid \varnothing\}$. Therefore, the proviso reduces spectacularly.

Proviso of $\mathcal{D}$:

(i) if $H = 0$, then $\mathrm{o}_L(G) = \mathrm{L}$;

(ii) if $G = 0$, then $\mathrm{o}_R(H) = \mathrm{R}$.

Another statement for the proviso Proviso(G, H) is that $\mathrm{o}(G) \geqslant \mathrm{o}(H)$; see [Larsson et al. 2021]. As for the answer to Q1, the best reference is [Larsson et al. 2021].

The decisive breakthrough related to the study of the algebra of $\mathcal{D}$, particularly the answer to Q2, occurred with the publication of [Dorbec et al. 2015]. They prove that atomic reversibility has an independent status: an atomic reversible option should be replaced by $* = \{0 \mid 0\}$ if it is the only winning option when played in isolation. Typically, an option like that cannot be entirely removed (as in normal play); in endgames, it may be the only way to win. For example, in the form $\{0, * \mid *\}$, the Left option $*$ cannot reverse out, unlike in normal play. In misère play, when playing $\{0, * \mid *\}$ in isolation, $*$ is the only winning choice for Left. Note also that when the atomic reversible option is the sole option, it cannot reverse out without the form ceasing to be dicotic. The exception is the form $\{* \mid *\} = 0$, subject to the reduction replacement by zero. It is the only case in which the game, after the reduction, does not cease to be dicotic, as both options of the form reverse out simultaneously. It is this simultaneity that explains the particular nature of this reduction and the reason why it is highlighted from the others. By these concepts, useful canonical forms are easy to obtain.

Regarding Q3 and the nature of invertible elements, it was proven in [Larsson et al. 2025a] that $\mathcal{D}$ satisfies the Conjugate Property. With the help of this fact, it was further demonstrated in [Fisher et al. 2022] that a dicotic canonical form G is invertible if and only if all followers G' satisfy $G' - G'$ is no $\mathcal{P}$-position.

2.3. *Dead-ending misère, $\mathcal{E}$.* Regarding the algebraic structure of $\mathcal{E}$, Milley and Renault [2013] established a first fundamental result, that the ends are invertible, with their inverses being their conjugates. For $n > 0$, interesting particular cases are the forms $\boldsymbol{n} = \{\boldsymbol{n-1} \mid \varnothing\}$ and $\overline{\boldsymbol{n}} = \{\varnothing \mid \overline{\boldsymbol{n-1}}\}$, corresponding to situations where Left (Right) has to make n consecutive moves with no alternative at their disposal. Naturally, $\boldsymbol{0} = 0 = \{\varnothing \mid \varnothing\}$ is the identity (Figure 1 illustrates the game trees of some of these forms).