GENERAL POST-NEWTONIAN ORBITAL EFFECTS

Orbital motions have always been used to test gravitational theories which, from time to time, have challenged the then-dominant paradigms. This book provides a unified treatment for calculating a wide variety of orbital effects due to general relativity and modified models of gravity, to its first and second post-Newtonian orders, in full generality. It gives explicit results valid for arbitrary orbital configurations and spin axes of the sources, without a priori simplifying assumptions on either the orbital eccentricity or inclination. These general results apply to a range of phenomena, from Earth's artificial satellites to the S-stars orbiting the supermassive black hole in the Galactic Centre to binary and triple pulsars, exoplanets, and interplanetary probes. Readers will become acquainted with working out a variety of orbital effects other than the time-honoured perihelion precession, designing their own space-based tests, performing effective sensitivity analyses, and assessing realistic error budgets.

LORENZO IORIO is qualified as Full Professor of Theoretical Physics and of Astrophysics at the Italian Ministry of University and Research. He earned his PhD from the University of Bari in 2001. His research focuses on gravitational physics, in particular, experimental/observational tests of general relativity and modified models of gravity. He is the author of more than 250 publications and is Editor-in-Chief of the journal *Universe*.

'This book provides a wonderful and very detailed guide for those interested in comparing observations with Einstein's theory and the many proposed alternatives. Written in a very readable and accessible manner, it is an indispensable guide to comparing theoretical gravitational predictions with the most recent data coming from celestial observations provided by satellites, space probes, and telescopes. I highly recommend it to anyone interested in a very practical handbook for comparing theory and observations.'

Jim Isenberg, Professor Emeritus, University of Oregon

'This scholarly book provides a comprehensive account of post-Newtonian orbital effects in gravitational systems. It is an authoritative contribution to modern relativistic celestial mechanics. Various gravitoelectric and gravitomagnetic effects of general relativity are treated in detail at the post-Newtonian level, while the last chapter of the book is devoted to modified gravity models. The presentation is clear and informative. This book is recommended to scientists working in astronomy and relativistic orbital mechanics.'

Bahram Mashhoon, Professor Emeritus of Physics, University of Missouri

'This is a self-contained text dealing with the main issue of any classical theory of gravity: orbital motion. The approach is very pedagogical. It is a precious toolkit to compare astronomical phenomenology with theories of gravity at any scale of astrophysical interest. The book is extremely useful for advanced undergraduate students as well as for PhD students in physics, astronomy, and mathematical physics. Furthermore, senior researchers working in the field can use it as a quick and comprehensive reference manual.'

Salvatore Capozziello, Full Professor, Università degli Studi di Napoli 'Federico II'

'Despite being mainly a theoretician, I find that this valuable book fills a gap in current literature since it sits at the interface between different fields which often do not communicate with one another. The approach and the methods developed in it can be straightforwardly extended from classical GR to alternative models of gravity, the orbital precessions of many of which are explicitly calculated. Interestingly, it explains how to potentially calculate the effect of any alternative gravity on several other observables such as astrometric angles, characteristic timescales, radial velocity, etc. I definitely recommend it to any serious student, researcher, and scholar involved in gravitational physics study.'

Sergei D. Odintsov, ICREA Research Professor, Institute of Space Sciences – CSIC, Spain

'An encyclopaedia of the 1pN and 2pN orbital effects, this book also explains approaches to testing GR in the said approximations. It is these practical applications which prove the great value of the post-Newtonian approximations, and which make this book an essential addition to the libraries of not only experts on GR and its applications, but also of the experts planning missions to giant planets. This excellent monograph provides a broad and up-to-date picture of post-Newtonian GR, as well as possible schemes of testing GR. I would recommend it to any graduate student or researcher working in the field of celestial mechanics and relativity.'

Michael Efroimsky, Astronomer, US Naval Observatory

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From Earth's Satellites to the Galactic Centre

LORENZO IORIO
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(Ministero dell'Istruzione e del Merito)

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Intentio vero nostra est manifestare ... ea, quae sunt, sicut sunt.

– Frederick II of Hohenstaufen (Holy Roman Emperor)
De arte venandi cum avibus

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