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Anthony T. Paxton
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Introduction

1.1 Momentum, Mass and Heat Transfer

A book should tell a story. ‘Begin at the beginning, and go on till you come to the end: then stop’, as the King of Hearts said, very gravely.[1] The story related here is how three phenomenological and historic laws of Newton, Fick and Fourier combine into a discipline as essential as thermodynamics and electrodynamics in the education of a scientist. The so called *transport phenomena* in fluids and solids have implications in subjects as diverse as plasma and astrophysics, planetary science, meteorology, microfluidics, hydraulics, energy systems, the design of ships and aircraft and the industrial processing of chemicals and engineering materials.

Figure 1.1 shows the structure of this book as a road map to the *fundamentals* of transport phenomena. At the origin, like a spider in the centre of her web, is the Reynolds transport theorem. At one level this is a trivial matter of book keeping; at another it is immensely powerful because when combined with the conservation laws upon which classical physics has been based since the nineteenth century, it leads to the principal formulas of the discipline. Conservation of mass leads to the equation of continuity (known in a qualitative form to Leonardo da Vinci). Conservation of linear momentum leads to the equations of motion of Augustin-Louis Cauchy, from which are obtained the famous Navier–Stokes equations[2] for an incompressible Newtonian fluid. Conservation of total energy leads from the transport theorem to the energy balance equation, from which may be derived the all-important heat conduction equation. In similar manner the diffusion equation evolves from mass continuity. These fundamental equations of fluid dynamics, heat conduction and diffusion require as input the phenomenological first order differential equations of Newton, Fourier and Fick; and they describe respectively the *transport* of momentum, heat and matter. Hence our subject has also been termed: heat, mass and momentum transfer (the title of many standard textbooks).

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[1] Lewis Carrol, ‘Alice’s Adventures in Wonderland’, chapter XII.

[2] Some authors write these in the singular. I write also Euler and Cauchy equations to emphasise that these are equations of a vector field, so indeed there are three of them: one for each component of the velocity vector.

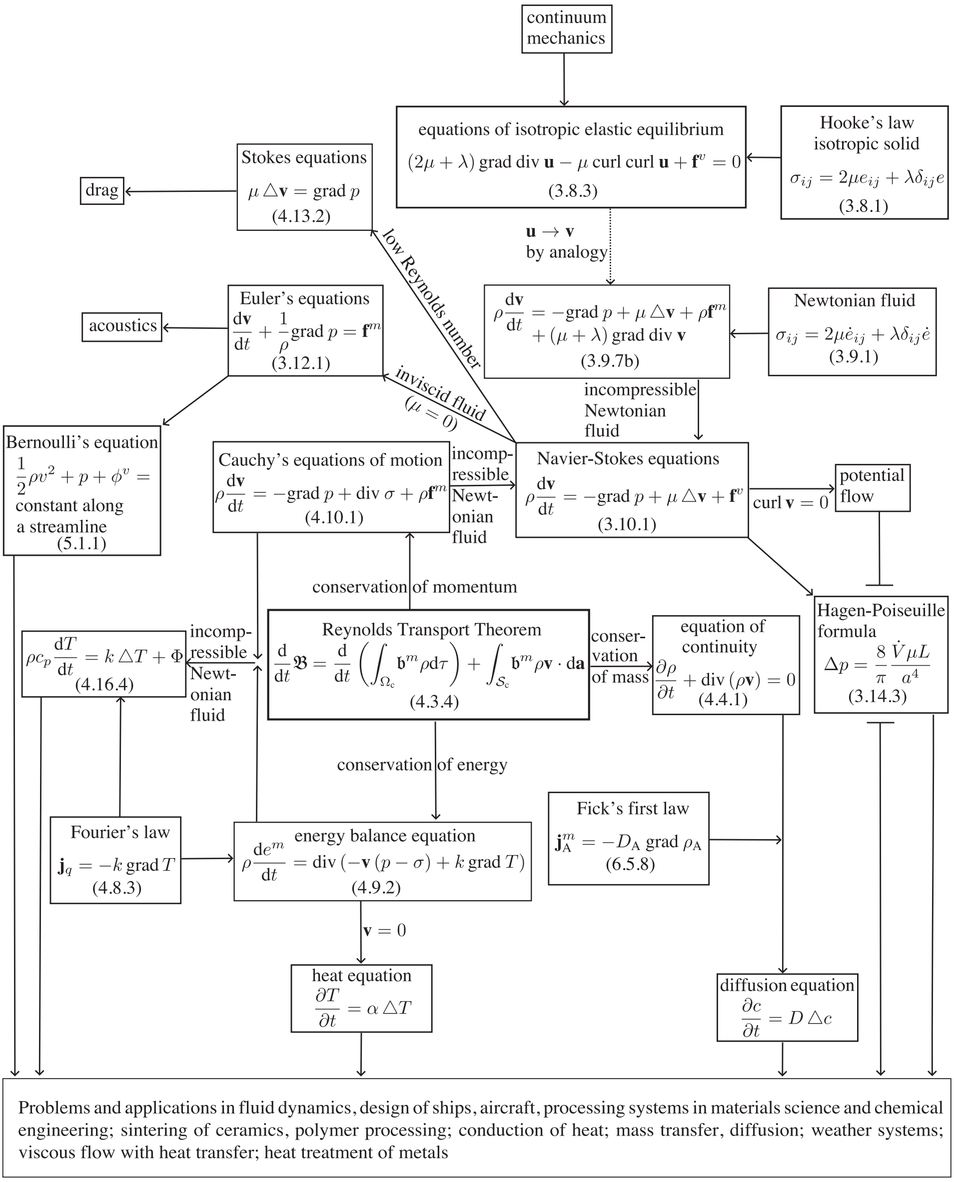

Figure 1.1 A flow chart of this book. This will seem incomprehensible at the first reading, but glance back frequently as the chapters are read. The first digit in each equation number is the chapter in which it appears; the second is the section number.

The story is not told here in a linear manner. Indeed the reader will come across a derivation of the continuity equation at least twice, and the same is true for the Navier–Stokes equations. It is a surprising fact that these may be obtained from

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the equations of equilibrium of an isotropic solid in continuum mechanics. For that reason, Chapter 3 is devoted to quite a digression into solid mechanics. Hooke's law – stress is proportional to strain – generalised to an isotropic solid contains two elastic constants:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda\delta_{ij}e\,,$$

where e is the dilatation, or volume strain – the *trace* of the tensor, e_{ij} – and μ and λ are *Lamé's constants*.[3]

If, by analogy, I replace strain with *strain rate* then I may write, as a proposition,

$$\sigma_{ij} = 2\mu\dot{e}_{ij} + \lambda\delta_{ij}\dot{e}\,,$$

where the over-dot denotes the time derivative, and μ and λ are now coefficients of *viscosity*. I have made the passage from elasticity of solids to the kinematics of a fluid. I will adopt the convention that this is the constitutive equation of a *Newtonian fluid*. We thereby retain the necessity of *two* coefficients of viscosity (a complication often avoided in the textbooks). If the fluid is, in addition, *incompressible* then

$$\dot{e} = \mathrm{div}\,\mathbf{v} = 0\,,$$

where $\mathbf{v}$ is the velocity field, and there results Newton's law of viscosity,[4]

$$\sigma_{xy} = \mu\frac{\mathrm{d}v_x}{\mathrm{d}y}\,,$$

which describes the transport of x-momentum in the y-direction.

While the intention is, as I say, to tell a story; on the other hand I have attempted to make each chapter self contained which is why certain results are obtained more than once, albeit from different starting points. The reader may choose not to read Chapter 3 at all, since its principal results for the purposes of the subject matter are Navier–Stokes equations, which are obtained from Reynolds's theorem in Chapter 4. However I have decided to introduce the Hagen–Poiseuille formula in this chapter, whereas I could easily have included it in the examples of Chapter 4, just so that you will need at least to glance at the last part of Chapter 3.

Chapter 4 possibly contains the principal substance of this book, particularly for the development of fluid mechanics from Reynolds's theorem; but the consequences of energy conservation are contained here also. A number of examples of practical application are found in this chapter, in particular Stokes's law for the drag of an object subject to fluid flow; Couette flow; and the Darcy law. It is in this chapter that we encounter one of the principal tools of the engineer: the *control volume*. Indeed Reynolds's theorem is in essence simply a book keeping or tally

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[3] Don't stress if you can't follow this, it will become clear in Chapter 3.

[4] To find out what happened to the 'two' you will have to wait until Chapter 6!

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of quantities that enter, that exit, and are created or destroyed in a chosen region in a flow problem. The control volume may be finite or infinitesimal. In this chapter will also be found a discussion of the two coefficients of viscosity, and also the various measures of pressure that we encounter. The chapter covers Prandtl's boundary layer; and closes with the derivation of the combined momentum and energy balance equations: at the centre left of Figure 1.1, Equation (4.16.4).

The 'ideal' or inviscid fluid is really a fiction, but it makes fluid dynamics so much easier. Many textbooks begin with this idealisation and introduce viscosity later on. If the flow is incompressible and *irrotational*, it yields to the mathematical beauty of potential theory, and it is remarkably useful in aeronautics even at air velocities approaching the speed of sound. Here, the inviscid fluid and Bernoulli's equation are the subject of Chapter 5, which also includes potential theory and a brief introduction to complex potentials. Chapter 6 is, perhaps, the most difficult and offers an introduction to the combined transport phenomena, which are needed when dealing with heat and mass transfer in a viscous fluid, with applications particularly in chemical engineering and processing in materials science. Chapters 7 and 8 cover solutions of the 'heat equation' in the contexts of mass transfer and heat conduction; Chapter 7 also includes a fairly detailed account of the fundamentals of diffusion in binary alloys at the atomic level.

To be perverse, I would strongly encourage the reader to start with Chapter 9! You will encounter a few results that will have been introduced in earlier chapters, but you will come to appreciate the vast power of dimensional analysis which has been a major tool of theoretical engineers ever since (or even before) Ludwig Prandtl came up with his theory of the boundary layer and his student Paul Blasius his equations, from dimensional arguments alone.

However, the book actually starts, in Chapter 2, with hydrostatics. Formally you may regard this in the context of Figure 1.1 as a consequence of Cauchy's equations in the absence of viscous stresses and if $\mathrm{d}v/\mathrm{d}t = 0$; but in practice hydrostatics is the study of fluids which are in some sense 'at rest'. Curiously this also includes the problems of the fixed and free vortices, and here we introduce the important property *vorticity* in the kinematics of fluid flow.

1.2 Prerequisites

This is a book about theoretical physics as much as anything else, and is hence couched in the language of that discipline: mathematics.[5] The reader is one who finds pleasure in mathematical manipulations leading to some quite non-trivial equations. Navier–Stokes are non-linear, second order partial differential equations and for that reason analytical solutions are elusive, and engineers resort to

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[5] I tell my students that mathematics is the language of physics in the same way that poetry is the language of epic (but that doesn't always go down very well).

computational fluid dynamics (a subject that is not covered here[6]). No result is stated here without proof and the reader, while comfortable with vector, and indeed tensor calculus, will find the origins of the operators grad, div and curl explained from just the basics of the differential calculus and elementary vector and tensor analysis.

The non-linearity is hidden in the formulas in Figure 1.1 in the innocent and innocuous looking total derivatives of the velocity field: $\mathrm{d}\mathbf{v}/\mathrm{d}t$. Some authors use an upper case 'D' to emphasise that this so called derivative following the motion is special in fluid dynamics, as indeed it is. Because students may find the concept hard to grasp, the total derivative is introduced separately in Chapters 3, 4 and 6, each from a different angle or viewpoint.

1.3 Notation

Although for completeness I include a table of symbols,[7] the intention is that this should not be necessary. I have followed strict rules so that any symbol should be unambiguous. An *extensive* thermodynamic property has a value that is proportional to the amount of substance such as volume, mass, momentum, total energy, total entropy. All *intensive* thermodynamic properties (those that do not depend on the amount of substance, such as pressure, temperature, density, specific heat) except for temperature, T, are given lower case symbols. If this is a 'specific' extensive quantity then a superscript m or v indicates a quantity per unit mass or unit volume, respectively.[8,9] Omission of the superscript implies the molar quantity. Extensive quantities are given upper case symbols. Vectors are written in bold; for example, $\mathbf{v}$ for velocity, while the corresponding non-bold symbol, v, means the magnitude of the vector. If a subscript is appended, for example v_x, this denotes the x-component of the vector, a dummy index, v_i for example, denotes the general i-component. While a scalar is formally a rank zero tensor, a vector is a rank one tensor. Rank two tensors, such as the Cauchy stress, σ_{ij}, carry two indices. The Einstein summation convention is used throughout. All this will be explained when we come to it.[10]

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[6] See Jonathan A. Danzig and Charles L. Tucker III, *Modeling in Materials Processing* (Cambridge University Press, New York, 2001).

[7] Not all symbols are listed, in cases that these are used only locally. Some symbols are used more than once: for example, γ is used as a ratio of specific heats, as specific weight, as a shear strain, surface tension and as activity coefficient. There will be no confusion as they are called upon only in their relevant places in the book.

[8] In equilibrium thermodynamics, a specific value of an extensive property makes sense only if the substance in question is *homogeneous*. See Kenneth Denbigh, *The Principles of Chemical Equilibrium*, 4th ed. (Cambridge University Press, Cambridge, 1981) Section 3.13; J. Willard Gibbs, *Collected Works*, Volume I (Longmans, Green, New York, 1931) page 62.

[9] However I use m rather than M for mass generally and ρ rather than m^v for mass density, which breaks my rule.

[10] But see, for example, William Prager, *Introduction to the Mechanics of Continua*, new edition (Dover Publications, New York, 1973).

I will use the symbols grad, div and curl, rather than using the vector $\boldsymbol{\nabla}$, and in addition to these three operators I will use the symbol $\triangle$ to denote the Laplace operator, which may act on scalar or vector fields. This is consistent with European usage, while in the UK and USA it is more usual in physics, if not mathematics, to use the symbol ∇^2; however this implies the 'square' of the grad vector, $\boldsymbol{\nabla}$, which the operator is only in a cartesian coordinate system. On the other hand, notably in Chapter 3, I shall use the 'comma' notation for derivatives of a field that stems originally from general relativity, but must be used with great care in a non-cartesian orthogonal system. Having said all that, while I have mostly avoided confining the equations to cartesian coordinates, most examples and exercises use the cartesian system. Do not confuse $\triangle$ with Δ, the latter being used to denote a change in some quantity, which we use in the Hagen–Poiseuille formula for the pressure drop, Δp, in flow along a pipe.

In this book the word *flux* is used strictly to mean the amount of some quantity that flows or is transported across a unit area in unit time. It will always be given a lower case $\mathbf{j}$, bold because it is a vector. In Chapter 6 I will introduce the 'net flux' which I denote with a fancy gothic $\mathfrak{j}$. The quantity that is transported will typically be heat or matter and this will be indicated with a subscript: q for heat and, say, A for molecular or atomic species, A. A superscript may also be employed to indicate in the latter case whether we are measuring the amount of substance in units of mass (superscript m) or number of moles (no superscript).

In this field as in others it is common to cast problems in terms of dimensionless, or *reduced* variables. In physics this can often lead to insights that are otherwise possibly concealed. Suppose you are interested in the Young's modulus, E, of metals as a function of temperature, T. Then you might construct a reduced temperature, $T' = T/T_m$, where T_m is a metal's melting point, and a reduced Young's modulus, $E' = E/E_{\mathrm{RT}}$ where E_{RT} is Young's modulus of the metal at room temperature. Then plotting E' against T' for a large range of metals between room temperature and the melting point may result in a graph in which all metals fall on the same curve. This will indicate that there is some underlying principle which is common to all the metals. In engineering, and especially in fluid mechanics, dimensionless numbers are frequently used to characterise a particular situation. This can be immensely powerful. We will find in Chapter 4, Section 4.13 that the Navier–Stokes equations can be written in terms of reduced variables, position, velocity, time and pressure. Then it emerges that if different experiments are compared, as long as the geometry and boundary conditions are similar then if in two or more instances the Reynolds numbers are identical then the velocity fields will be identical. This is true if gravity can be disregarded; if not, then the Froude and Reynolds numbers must be identical across the two experiments. This so called *dynamical similarity* is exploited in the simulations of ships, using smaller prototypes or in the design of racing cars in a view to reduce drag by the well known use of wind tunnels. In regard to notation, in this book all reduced variables will be indicated with a ‘prime’ superscript (as above with T' and E'). There is not always a unique or obvious way to non-dimensionalise a variable; therefore if there are more ways to do this, to avoid ambiguity, I use a double prime or a ‘star’ (which is the most frequently used notation in the textbooks) in Section 4.13. Otherwise I may use a single prime on two or more reduced variables, say a drag coefficient, which are not the same. There will be no possibility of confusion as all will be clear from the text.

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Hydrostatics

2.1 Statics Kinematics Dynamics

Hydrostatics, as the name implies, is the subject of fluids at rest. Of course 'rest' is a relative notion, so it may be as well to replace this with 'rest within some inertial or non-inertial frame (such as the rotating earth)'. It is usual also to restrict *hydrostatics* to fluids that are subject only to body forces. By *body force* we imply those forces which arise from action at a distance,[1] such as gravity or electrostatic force. In this class I may also place centrifugal and Coriolis forces, although these are often called *inertial*, or *fictitious*, since they arise from acceleration due to motion in a rotating frame. Later I will introduce contact forces, such as mechanical stress which is transmitted by *tractions* acting locally either at a surface or internally across a notional cut in the body. Internal stresses have their origins in the interatomic forces that maintain the integrity of a body. If we deal with problems in which a velocity field is *given*, or in which we deal with the motion or distortion of a body without concern about the forces that give rise to it, then we call this a problem in *kinematics*.[2] Further on we shall move to *dynamics* in which we add inertial resisting forces in accord with the principle of d'Alembert that lead us to *equations of motion* – in particular the Navier–Stokes equations. We will see a mathematical definition of hydrostatics, in Section 4.10 obtained from the equations of motion in the case there are no viscous stresses and no time dependence of the velocity field.

2.2 Pressure

Pressure acts equally in all directions. The density of a liquid depends only on temperature. The liquid is *incompressible*. However, as you are well aware, the

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[1] or from a *field* in the more post nineteenth century way of thinking,

[2] The distinction between kinematics and dynamics is expressed rather well by Leopold Infeld: kinematics is the science of *describing* motion, dynamics of *predicting* it.

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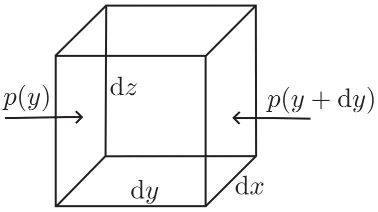

Figure 2.1 An elemental cuboid of fluid

pressure will vary with depth, for example in the ocean. A gas is compressible so both its density and its pressure will vary, say, with the height above the surface of the earth.

Imagine an infinitesimal volume element of fluid, $d\tau = dx\,dy\,dz$, in the sense of the integral calculus. This is sketched in Figure 2.1. The cuboid experiences a *body force*, $\mathbf{f}^v$, *per unit volume*. In this whole text we will only admit gravity as the body force. The cuboid may also experience forces due to the pressure being infinitesimally different at opposite faces. For example, Figure 2.1 shows the pressure, p, acting at the face whose normal is parallel to the y-axis, and the pressure is not necessarily the same at y and at $y + dy$. The total force in the y-direction then comprises three contributions.

1. Body force per unit volume times volume:

$$f_y^v\,d\tau = f_y^v\,dx\,dy\,dz.$$

2. Force acting on the left hand face, pressure times area:

$$p(y)dx\,dz.$$

3. Force acting on the right hand face, pressure times area:

$$-p(y + dy)dx\,dz.$$

I will use the definition of a partial differential quotient,

$$\frac{p(x, y + dy, z) - p(x, y, z)}{dy} = \frac{\partial p}{\partial y},$$

to state the total y-component of the force per unit volume:

$$-\frac{\partial p}{\partial y} + f_y^v.$$

By identical arguments the x- and z-components of the force per unit volume are

$$-\frac{\partial p}{\partial x} + f_x^v \quad \text{and} \quad -\frac{\partial p}{\partial z} + f_z^v.$$

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When we have three equations for the x, y and z components of a vector, we multiply them respectively by the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ along the x, y and z axes and we get the vector in its cartesian components:

$$\left(-\frac{\partial p}{\partial x} + f_y^v\right)\hat{\mathbf{x}} + \left(-\frac{\partial p}{\partial y} + f_y^v\right)\hat{\mathbf{y}} + \left(-\frac{\partial p}{\partial z} + f_z^v\right)\hat{\mathbf{z}}.$$

The expression,

$$\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) p,$$

is called the *gradient of the scalar field*, p. In cartesian coordinates the differential operator in parentheses is given the vector symbol, $\boldsymbol{\nabla}$. More generally it is called ‘grad’ and the previous expression is grad p.[3]

In equilibrium the total force on any element of the fluid has to be zero, so I end up with

$$\text{grad}\, p = \mathbf{f}^v. \tag{2.2.1}$$

For equilibrium to be possible, if the left hand side is the gradient of a scalar, then so must the right hand side be. So there must exist a potential energy function per unit volume, ϕ^v, and I will express this as

$$\mathbf{f}^v = -\text{grad}\, \phi^v.$$

The minus sign is to some extent conventional, but it is consistent with the notion that if there is a potential energy which is a function, say, of the position of a particle along the x-direction then the force the particle experiences is *minus* the gradient of the potential energy so the particle rolls *down* the energy slope and ends up, say, at the bottom of a potential well; this is the case for a simple pendulum, or a ball bearing in a soup bowl. I now can write (2.2.1) as

$$\text{grad}\,(p + \phi^v) = 0,$$

and it follows that

$$p + \phi^v = \text{constant}. \tag{2.2.2}$$

Now the only body force we consider is gravity and so ϕ^v is the gravitational potential energy per unit volume of the body in question. If a body has a mass m

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[3] When $\boldsymbol{\nabla}$ acts on a scalar, as here, the result is a vector; $\boldsymbol{\nabla}$ can also act on a vector to produce a scalar (this is called contraction); if $\mathbf{v} = \hat{\mathbf{x}}v_x + \hat{\mathbf{y}}v_y + \hat{\mathbf{z}}v_z$, we define

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z},$$

which is called the ‘divergence’ of the vector $\mathbf{v}$. Again, to be more generally applicable in any orthogonal coordinate system (for example spherical or cylindrical in addition to cartesian) we will write this as div $\mathbf{v}$, which avoids the inclination to think of $\boldsymbol{\nabla}$ as a vector, which it is only in a cartesian system of coordinates.