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1
Banach Spaces

The foundations of modern Analysis were laid in the early decades of the twentieth century, through the work of Maurice Fréchet, Ivar Fredholm, David Hilbert, Henri Lebesgue, Frigyes Riesz, and many others. These authors realised that it is fruitful to study linear operations in a setting of abstract spaces endowed with further structure to accommodate the notions of convergence and continuity. This led to the introduction of abstract topological and metric spaces and, when combined with linearity, of topological vector spaces, Hilbert spaces, and Banach spaces. Since then, these spaces have played a prominent role in all branches of Analysis.

The main impetus came from the study of ordinary and partial differential equations where linearity is an essential ingredient, as evidenced by the linearity of the main operations involved: point evaluations, integrals, and derivatives. It was discovered that many theorems known at the time, such as existence and uniqueness results for ordinary differential equations and the Fredholm alternative for integral equations, can be conveniently abstracted into general theorems about linear operators in infinite-dimensional spaces of functions.


Stefan Banach, 1898–1945

A second source of inspiration was the discovery, in the 1920s by John von Neumann, that the – at that time brand new – theory of Quantum Mechanics can be put on a solid mathematical foundation by means of the spectral theory of selfadjoint operators on Hilbert spaces. It was not until the 1930s that these two lines of mathematical thinking were brought together in the theory of Banach spaces, named after its creator Stefan Banach (although this class of spaces was also discovered, independently and about the same time, by Norbert Wiener). This theory provides a unified perspective on Hilbert spaces

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and the various spaces of functions encountered in Analysis, including the spaces $C(K)$ of continuous functions and the spaces $L^p(\Omega)$ of Lebesgue integrable functions.

1.1 Banach Spaces

The aim of the present chapter is to introduce the class of Banach spaces and discuss some elementary properties of these spaces. The main classical examples are only briefly mentioned here; a more detailed treatment is deferred to the next two chapters. Much of the general theory applies to both the real and complex scalar field. Whenever this applies, the symbol $\mathbb{K}$ is used to denote the scalar field, which is $\mathbb{R}$ in the case of real vector spaces and $\mathbb{C}$ in the case of complex vector spaces.

1.1.a Definition and General Properties

Definition 1.1 (Norms)**.** A *normed space* is a pair $(X, \|\cdot\|)$, where X is a vector space over $\mathbb{K}$ and $\|\cdot\| : X \to [0,\infty)$ is a *norm*, that is, a mapping with the following properties:

(i) $\|x\| = 0$ implies $x = 0$;
(ii) $\|cx\| = |c|\,\|x\|$ for all $c \in \mathbb{K}$ and $x \in X$;
(iii) $\|x + x'\| \leqslant \|x\| + \|x'\|$ for all $x, x' \in X$.

When the norm $\|\cdot\|$ is understood we simply write X instead of $(X, \|\cdot\|)$. If we wish to emphasise the role of X we write $\|\cdot\|_X$ instead of $\|\cdot\|$.

The properties (ii) and (iii) are referred to as *scalar homogeneity* and the *triangle inequality*. The triangle inequality implies that every normed space is a metric space, with distance function

$$d(x,y) := \|x - y\|.$$

This observation allows us to introduce notions such as openness, closedness, compactness, denseness, limits, convergence, completeness, and continuity in the context of normed spaces by carrying them over from the theory of metric spaces. For instance, a sequence $(x_n)_{n\geqslant 1}$ in X is said to *converge* if there exists an element $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. This element, if it exists, is unique and is called the *limit* of the sequence $(x_n)_{n\geqslant 1}$. We then write $\lim_{n\to\infty} x_n = x$ or simply '$x_n \to x$ as $n \to \infty$'.

The triangle inequality (ii) implies both $\|x\| - \|x'\| \leqslant \|x - x'\|$ and $\|x'\| - \|x\| \leqslant \|x' - x\|$. Since $\|x' - x\| = \|(-1)\cdot(x - x')\| = \|x - x'\|$ by scalar homogeneity, we obtain the *reverse triangle inequality*

$$\big|\|x\| - \|x'\|\big| \leqslant \|x - x'\|.$$

It shows that taking norms $x \mapsto \|x\|$ is a continuous operation.

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If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x'_n = x'$ in X and $c \in \mathbb{K}$ is a scalar, then $\|cx_n - cx\| = \|c(x_n - x)\| = |c|\|x_n - x\|$ implies

$$\lim_{n\to\infty} \|cx_n - cx\| = 0.$$

Likewise, $\|(x_n + x'_n) - (x + x')\| = \|(x_n - x) + (x'_n - x')\| \leqslant \|x_n - x\| + \|x'_n - x'\|$ implies

$$\lim_{n\to\infty} \|(x_n + x'_n) - (x + x')\| = 0.$$

This proves sequential continuity, and hence continuity, of the vector space operations.

Throughout this work we use the notation

$$B(x_0; r) := \{x \in X : \|x - x_0\| < r\}$$

for the *open ball* centred at $x_0 \in X$ with radius $r > 0$, and

$$\overline{B}(x_0; r) := \{x \in X : \|x - x_0\| \leqslant r\}$$

for the corresponding *closed ball*. The *open unit ball* and *closed unit ball* are the balls

$$B_X := B(0; 1) = \{x \in X : \|x\| < 1\}, \quad \overline{B}_X := \overline{B}(0; 1) = \{x \in X : \|x\| \leqslant 1\}.$$

Definition 1.2 (Banach spaces)**.** A *Banach space* is a complete normed space.

Thus a Banach space is a normed space X in which every Cauchy sequence is convergent, that is, $\lim_{m,n\to\infty} \|x_n - x_m\| = 0$ implies the existence of an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

The following proposition gives a necessary and sufficient condition for a normed space to be a Banach space. We need the following terminology. Given a sequence $(x_n)_{n\geqslant 1}$ in a normed space X, the sum $\sum_{n\geqslant 1} x_n$ is said to be *convergent* if there exists $x \in X$ such that

$$\lim_{N\to\infty} \Big\| x - \sum_{n=1}^{N} x_n \Big\| = 0.$$

The sum $\sum_{n\geqslant 1} x_n$ is said to be *absolutely convergent* if $\sum_{n\geqslant 1} \|x_n\| < \infty$.

Proposition 1.3. *A normed space X is a Banach space if and only if every absolutely convergent sum in X converges in X.*

Proof ‘Only if’: Suppose that X is complete and let $\sum_{n\geqslant 1} x_n$ be absolutely convergent. Then the sequence of partial sums $(\sum_{j=1}^{n} x_j)_{n\geqslant 1}$ is a Cauchy sequence, for if $n > m$ the triangle inequality implies

$$\Big\| \sum_{j=1}^{n} x_j - \sum_{j=1}^{m} x_j \Big\| = \Big\| \sum_{j=m+1}^{n} x_j \Big\| \leqslant \sum_{j=m+1}^{n} \|x_j\|,$$

which tends to 0 as $m, n \to \infty$. Hence, by completeness, the sum $\sum_{n\geqslant 1} x_n$ converges.

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'If': Suppose that every absolutely convergent sum in X converges in X, and let $(x_n)_{n\geqslant 1}$ be a Cauchy sequence in X. We must prove that $(x_n)_{n\geqslant 1}$ converges in X.

Choose indices $n_1 < n_2 < \dots$ in such a way that $\|x_i - x_j\| < \frac{1}{2^k}$ for all $i,j \geqslant n_k$, $k = 1,2,\dots$ The sum $x_{n_1} + \sum_{k\geqslant 1}(x_{n_{k+1}} - x_{n_k})$ is absolutely convergent since

$$\sum_{k\geqslant 1} \|x_{n_{k+1}} - x_{n_k}\| \leqslant \sum_{k\geqslant 1} \frac{1}{2^k} < \infty.$$

By assumption it converges to some $x \in X$. Then, by cancellation,

$$x = \lim_{m\to\infty} \Big(x_{n_1} + \sum_{k=1}^{m} (x_{n_{k+1}} - x_{n_k}) \Big) = \lim_{m\to\infty} x_{n_{m+1}},$$

and therefore the subsequence $(x_{n_m})_{m\geqslant 1}$ is convergent, with limit x. To see that $(x_n)_{n\geqslant 1}$ converges to x, we note that

$$\|x_m - x\| \leqslant \|x_m - x_{n_m}\| + \|x_{n_m} - x\| \to 0$$

as $m \to \infty$ (the first term since we started from a Cauchy sequence and the second term by what we just proved). $\square$

The next theorem asserts that every normed space can be completed to a Banach space. For the rigorous formulation of this result we need the following terminology.

Definition 1.4 (Isometries)**.** A linear mapping T from a normed space X into a normed space Y is said to be an *isometry* if it preserves norms. A normed space X is *isometrically contained* in a normed space Y if there exists an isometry from X into Y.

Theorem 1.5 (Completion)**.** *Let X be a normed space. Then:*

(1) *there exists a Banach space $\overline{X}$ containing X isometrically as a dense subspace;*
(2) *the space $\overline{X}$ is unique up to isometry in the following sense: If X is isometrically contained as a dense subspace in the Banach spaces $\overline{X}$ and $\overline{\overline{X}}$, then the identity mapping on X has a unique extension to an isometry from $\overline{X}$ onto $\overline{\overline{X}}$.*

Proof As a metric space, $X = (X,d)$ has a completion $\overline{X} = (\overline{X},\overline{d})$ by Theorem D.6. We prove that $\overline{X}$ is a Banach space in a natural way, with a norm $\|\cdot\|_{\overline{X}}$ such that $\overline{d}(x,x') = \|x - x'\|_{\overline{X}}$. The properties (1) and (2) then follow from the corresponding assertions for metric spaces.

Recall that the completion $\overline{X}$ of X, as a metric space, is defined as the set of all equivalence classes of Cauchy sequences in X, declaring the Cauchy sequences $(x_n)_{n\geqslant 1}$ and $(x_n')_{n\geqslant 1}$ to be equivalent if $\lim_{n\to\infty} d(x_n,x_n') = \lim_{n\to\infty} \|x_n - x_n'\| = 0$. The space $\overline{X}$ is a vector space under the scalar multiplication

$$c[(x_n)_{n\geqslant 1}] := [c(x_n)_{n\geqslant 1}]$$

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and addition

$$[(x_n)_{n\geqslant 1}] + [(x'_n)_{n\geqslant 1}] := [(x_n + x'_n)_{n\geqslant 1}],$$

where the brackets denote the equivalence class.

If $(x_n)_{n\geqslant 1}$ is a Cauchy sequence in X, the reverse triangle inequality implies that the nonnegative sequence $(\|x_n\|)_{n\geqslant 1}$ is Cauchy, and hence convergent by the completeness of the real numbers. We now define a norm on $\overline{X}$ by

$$\|[(x_n)_{n\geqslant 1}]\|_{\overline{X}} := \lim_{n\to\infty} \|x_n\|.$$

Denoting by $\overline{d}$ the metric on $\overline{X}$ given by $\overline{d}(x,x') := \lim_{n\to\infty} d(x_n,x'_n)$, where $x = (x_n)_{n\geqslant 1}$ and $x' = (x'_n)_{n\geqslant 1}$, it is clear that $\overline{d}(x,x') = \|x - x'\|_{\overline{X}}$. □

1.1.b Subspaces, Quotients, and Direct Sums

Several abstract constructions enable us to create new Banach spaces from given ones. We take a brief look at the three most basic constructions, namely, passing to closed subspaces and quotients and taking direct sums.

Subspaces A subspace Y of a normed space X is a normed space with respect to the norm inherited from X. A subspace Y of a Banach space X is a Banach space with respect to the norm inherited from X if and only if Y is closed in X.

To prove the 'if' part, suppose that $(y_n)_{n\geqslant 1}$ is a Cauchy sequence in the closed subspace Y of a Banach space X. Then it has a limit in X, by the completeness of X, and this limit belongs to Y, by the closedness of Y. The proof of the 'only if' part is equally simple and does not require X to be complete. If $(y_n)_{n\geqslant 1}$ is a sequence in the complete subspace Y such that $y_n \to x$ in X, then $(y_n)_{n\geqslant 1}$ is a Cauchy sequence in X, hence also in Y, and therefore it has a limit y in Y, by the completeness of Y. Since $(y_n)_{n\geqslant 1}$ also converges to y in X, it follows that $y = x$ and therefore $x \in Y$.

Quotients If Y is a closed subspace of a Banach space X, the quotient space X/Y can be endowed with a norm by

$$\|[x]\| := \inf_{y\in Y} \|x - y\|,$$

where for brevity we write $[x] := x + Y$ for the equivalence class of x modulo Y. Let us check that this indeed defines a norm. If $\|[x]\| = 0$, then there is a sequence $(y_n)_{n\geqslant 1}$ in Y such that $\|x - y_n\| < \frac{1}{n}$ for all $n \geqslant 1$. Then

$$\|y_n - y_m\| \leqslant \|y_n - x\| + \|x - y_m\| < \frac{1}{n} + \frac{1}{m},$$

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so $(y_n)_{n\geqslant 1}$ is a Cauchy sequence in X. It has a limit $y \in X$ since X is complete, and we have $y \in Y$ since Y is closed. Then $\|x-y\| = \lim_{n\to\infty} \|x-y_n\| = 0$, so $x = y$. This implies that $[x] = [y] = [0]$, the zero element of X/Y. The identity $\|c[x]\| = |c|\|[x]\|$ is trivially verified, and so is the triangle inequality.

To see that the normed space X/Y is complete we use the completeness of X and Proposition 1.3. If $\sum_{n\geqslant 1} \|[x_n]\| < \infty$ and the $y_n \in Y$ are such that $\|x_n - y_n\| \leqslant \|[x_n]\| + \frac{1}{n^2}$, the proposition implies that $\sum_{n\geqslant 1}(y_n - x_n)$ converges in X, say to x. Then, for all $N \geqslant 1$,

$$\Big\|[x] - \sum_{n=1}^{N} [x_n]\Big\| = \Big\|\Big[x - \sum_{n=1}^{N} x_n\Big]\Big\| \leqslant \Big\|x - \sum_{n=1}^{N} x_n + \sum_{n=1}^{N} y_n\Big\| = \Big\|x - \Big(\sum_{n=1}^{N} x_n - y_n\Big)\Big\|.$$

As $N \to \infty$, the right-hand side tends to 0 and therefore $\lim_{N\to\infty} \sum_{n=1}^{N} [x_n] = [x]$ in X/Y.

Direct Sums A *product norm* on a finite cartesian product $X = X_1 \times \cdots \times X_N$ of normed spaces is a norm $\|\cdot\|$ satisfying

$$\|(0,\ldots,0,\underbrace{x_n}_{n-\text{th}},0,\ldots,0)\| = \|x_n\| \leqslant \|(x_1,\ldots,x_N)\|$$

for all $x = (x_1,\ldots,x_N) \in X$ and $n = 1,\ldots,N$. For instance, every norm $|\cdot|$ on $\mathbb{K}^N$ assigning norm one to the standard unit vectors induces a product norm on X by the formula

$$\|(x_1,\ldots,x_N)\| := \big|(\|x_1\|,\ldots,\|x_N\|)\big|. \tag{1.1}$$

As a normed space endowed with a product norm, the cartesian product will be denoted

$$X = X_1 \oplus \cdots \oplus X_N$$

and called a *direct sum* of $X_1,\ldots,X_N$. If every X_n is a Banach space, then the normed space X is a Banach space. Indeed, from

$$\|x\| = \Big\|\sum_{n=1}^{N} (0,\ldots,0,x_n,0,\ldots,0)\Big\| \leqslant \sum_{n=1}^{N} \|x_n\| \leqslant N\|x\| \tag{1.2}$$

we see that a sequence $(x^{(k)})_{k\geqslant 1}$ in X is Cauchy if and only if all its coordinate sequences $(x_n^{(k)})_{k\geqslant 1}$ are Cauchy. If the spaces X_n are complete, these coordinate sequences have limits x_n in X_n, and these limits serve as the coordinates of an element $x = (x_1,\ldots,x_N)$ in X which is the limit of the sequence $(x^{(k)})_{k\geqslant 1}$.

1.1.c First Examples

The purpose of this brief section is to present a first catalogue of Banach spaces. The presentation is not self-contained; the examples will be revisited in more detail in the next chapter, where the relevant terminology is introduced and proofs are given.

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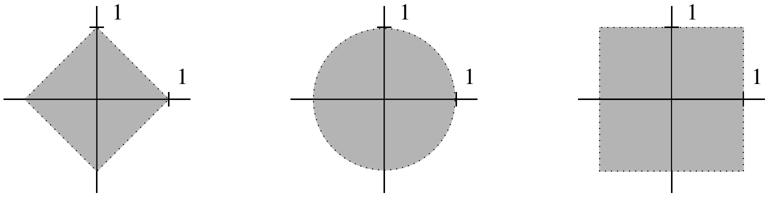

Figure 1.1 The open unit balls of $\mathbb{R}^2$ with respect to the norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$.

Example 1.6 (Euclidean spaces)**.** On $\mathbb{K}^d$ we may consider the euclidean norm

$$\|a\|_2 := \Big(\sum_{j=1}^d |a_j|^2\Big)^{1/2},$$

and more generally the p-norms

$$\|a\|_p := \Big(\sum_{j=1}^d |a_j|^p\Big)^{1/p}, \quad 1 \leqslant p < \infty,$$

as well as the supremum norm

$$\|a\|_\infty := \sup_{1\leqslant j\leqslant d} |a_j|.$$

It is not immediately obvious that the p-norms are indeed norms; the triangle inequality $\|a+b\|_p \leqslant \|a\|_p + \|b\|_p$ will be proved in the next chapter. It is an easy matter to check that the above norms are all *equivalent* in the sense defined in Section 1.3. *In what follows the euclidean norm of an element $x \in \mathbb{K}^d$ is denoted by* $|x|$ instead of the more cumbersome $\|x\|_2$.

Example 1.7 (Sequence spaces)**.** Thinking of elements of $\mathbb{K}^d$ as finite sequences, the preceding example may be generalised to infinite sequences as follows. For $1 \leqslant p < \infty$ the space ℓ^p is defined as the space of all scalar sequences $a = (a_k)_{k\geqslant 1}$ satisfying

$$\|a\|_p := \Big(\sum_{k\geqslant 1} |a_k|^p\Big)^{1/p} < \infty.$$

The mapping $a \mapsto \|a\|_p$ is a norm which turns ℓ^p into a Banach space. The space ℓ^∞ of all bounded scalar sequences $a = (a_k)_{k\geqslant 1}$ is a Banach space with respect to the norm

$$\|a\|_\infty := \sup_{k\geqslant 1} |a_k| < \infty.$$

The space c_0 consisting of all bounded scalar sequences $a = (a_k)_{k\geqslant 1}$ satisfying

$$\lim_{k\to\infty} a_k = 0$$

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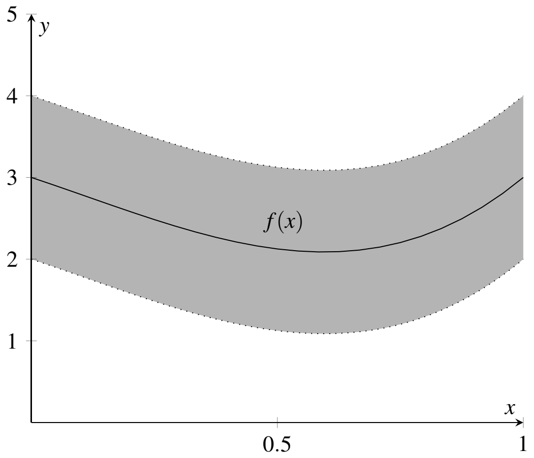

Figure 1.2 The open ball $B(f;1)$ in $C[0,1]$ consists of all functions in $C[0,1]$ whose graph lies inside the shaded area.

is a closed subspace of ℓ^∞. As such it is a Banach space in its own right.

Example 1.8 (Spaces of continuous functions)**.** Let K be a compact topological space. The space $C(K)$ of all continuous functions $f : K \to \mathbb{K}$ is a Banach space with respect to the supremum norm

$$\|f\|_\infty := \sup_{x\in K} |f(x)|.$$

This norm captures the notion of uniform convergence: for functions in $C(K)$ we have $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$ if and only if $\lim_{n\to\infty} f_n = f$ uniformly.

Example 1.9 (Spaces of integrable functions)**.** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. For $1 \leqslant p < \infty$, the space $L^p(\Omega)$ consisting of all measurable functions $f : \Omega \to \mathbb{K}$ such that

$$\|f\|_p := \Big(\int_\Omega |f|^p \,\mathrm{d}\mu\Big)^{1/p} < \infty,$$

identifying functions that are equal μ-almost everywhere, is a Banach space with respect to the norm $\|\cdot\|_p$. The space $L^\infty(\Omega)$ consisting of all measurable and μ-essentially bounded functions $f : \Omega \to \mathbb{K}$, identifying functions that are equal μ-almost everywhere, is a Banach space with respect to the norm given by the μ-essential supremum

$$\|f\|_\infty := \mu\text{-}\operatorname*{ess\,sup}_{\omega\in\Omega} |f(\omega)| := \inf\big\{r > 0 : |f| \leqslant r \ \mu\text{-almost everywhere}\big\}.$$

Example 1.10 (Spaces of measures)**.** Let $(\Omega, \mathscr{F})$ be a measurable space. The space

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$M(\Omega)$ consisting of all $\mathbb{K}$-valued measures of bounded variation on $(\Omega, \mathscr{F})$ is a Banach space with respect to the variation norm

$$\|\mu\| := |\mu|(\Omega) := \sup_{\mathscr{A} \in \mathbb{F}} \sum_{A \in \mathscr{A}} |\mu(A)|,$$

where $\mathbb{F}$ denotes the set of all finite collections of pairwise disjoint sets in $\mathscr{F}$.

Example 1.11 (Hilbert spaces)**.** A *Hilbert space* is an inner product space $(H, (\cdot|\cdot))$ that is complete with respect to the norm

$$\|h\| := (h|h)^{1/2}.$$

Examples include the spaces $\mathbb{K}^d$ with the euclidean norm, ℓ^2, and the spaces $L^2(\Omega)$. Further examples will be given in later chapters.

1.1.d Separability

Most Banach spaces of interest in Analysis are *infinite-dimensional* in the sense that they do not have a finite spanning set. In this context the following definition is often useful.

Definition 1.12 (Separability)**.** A normed space is called *separable* if it contains a countable set whose linear span is dense.

Proposition 1.13. *A normed space X is separable if and only if X contains a countable dense set.*

Proof The 'if' part is trivial. To prove the 'only if' part, let $(x_n)_{n\geqslant 1}$ have dense span in X. Let Q be a countable dense set in $\mathbb{K}$ (for example, one could take $Q = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $Q = \mathbb{Q} + i\mathbb{Q}$ if $K = \mathbb{C}$). Then the set of all Q-linear combinations of the x_n, that is, all linear combinations involving coefficients from Q, is dense in X. □

Finite-dimensional spaces, the sequence spaces c_0 and ℓ^p with $1 \leqslant p < \infty$, the spaces $C(K)$ with K compact metric, and $L^p(D)$ with $1 \leqslant p < \infty$ and $D \subseteq \mathbb{R}^d$ open, are separable. The separability of $C(K)$ and $L^p(D)$ follows from the results proved in the next chapter.

1.2 Bounded Operators

Having introduced normed spaces and Banach spaces, we now introduce a class of linear operators acting between them which interact with the norm in a meaningful way.

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1.2.a Definition and General Properties

Let X and Y be normed spaces.

Definition 1.14 (Bounded operators)**.** A linear operator $T : X \to Y$ is *bounded* if there exists a finite constant $C \geqslant 0$ such that

$$\|Tx\| \leqslant C\|x\|, \quad x \in X.$$

Here, and in the rest of this work, we write Tx instead of the more cumbersome $T(x)$. A *bounded operator* is a linear operator that is bounded.

The infimum C_T of all admissible constants C in Definition 1.14 is itself admissible. Thus C_T is the least admissible constant. We claim that it equals the number

$$\|T\| := \sup_{\|x\|\leqslant 1} \|Tx\|.$$

To see this, let C be an admissible constant in Definition 1.14, that is, we assume that $\|Tx\| \leqslant C\|x\|$ for all $x \in X$. Then $\|T\| = \sup_{\|x\|\leqslant 1} \|Tx\| \leqslant C$. This being true for all admissible constants C, it follows that $\|T\| \leqslant C_T$. The opposite inequality $C_T \leqslant \|T\|$ follows by observing that for all $x \in X$ we have

$$\|Tx\| \leqslant \|T\|\|x\|,$$

which means that $\|T\|$ an admissible constant. This inequality is trivial for $x = 0$, and for $x \neq 0$ it follows from scalar homogeneity, the linearity of T and the definition of the number $\|T\|$:

$$\|Tx\| = \Big\|\frac{1}{\|x\|}Tx\Big\|\|x\| = \Big\|T\frac{x}{\|x\|}\Big\|\|x\| \leqslant \|T\|\|x\|.$$

Proposition 1.15. *For a linear operator $T : X \to Y$ the following assertions are equivalent:*

(1) *T is bounded;*
(2) *T is continuous;*
(3) *T is continuous at some point $x_0 \in X$.*

Proof The implication (1)⇒(2) follows from

$$\|Tx - Tx'\| = \|T(x - x')\| \leqslant \|T\|\|x - x'\|$$

and the implication (2)⇒(3) is trivial. To prove the implication (3)⇒(1), suppose that T is continuous at x_0. Then there exists a $\delta > 0$ such that $\|x_0 - y\| < \delta$ implies $\|Tx_0 - Ty\| < 1$. Since every $x \in X$ with $\|x\| < \delta$ is of the form $x = x_0 - y$ with $\|x_0 - y\| < \delta$ (take $y = x_0 - x$) and T is linear, it follows that $\|x\| < \delta$ implies $\|Tx\| < 1$. By scalar homogeneity and the linearity of T we may scale both sides with a factor δ, and obtain

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