1 Propositional Quantifiers

1.1 Introduction

The simplest languages of formal logic are propositional. These languages provide sentential letters and connectives with which we can represent, for example, the conditional structure of a sentence like the following:

(1) If you invented the smiley, then I am the pope.

Letting s stand for you inventing the smiley, p for me being the pope, and using $\rightarrow$ to represent the conditional in (1), this can be formalized as follows:

(2) $s \rightarrow p$

Other familiar sentential connectives, such as negation, conjunction, disjunction, and the biconditional can be treated similarly, for which I will use the symbols $\neg$, $\wedge$, $\vee$, and $\leftrightarrow$ respectively.

It is often supposed that the connectives just mentioned are truth-functional. For example, the truth-value of $\neg p$ is plausibly determined by the truth-value of p: $\neg p$ is true just in case p is false, and $\neg p$ is false just in case p is true. But many applications of propositional logics across different disciplines require sentential connectives which are not truth-functional. Standard examples involve modal and epistemic notions, such as the following two:

(3) Necessarily, Jupiter is a planet.
(4) Kushim believes correctly that barley is a grain.

Necessity and belief are not truth-functional: some but not all truths are necessary, and some but not all truths (and falsehoods) may be believed by a given agent. Nevertheless, it is straightforward to extend the language of propositional logic to capture these statements as well. For the first, it suffices to introduce a sentential operator $\Box$ for necessity, and for the second, a sentential operator B_k for being believed by Kushim:

(5) $\Box r$
(6) $B_k g \wedge g$

In (5) and (6), r stands for Jupiter being a planet, and g for barley being a grain. The study of logical systems with sentential operators which are not truth-functional is known as *modal logic*; see, for example, Hughes and Cresswell (1996) and Blackburn et al. (2001).

All three of the examples just mentioned are particular, as opposed to general. For example, (4) attributes to Kushim a particular (correct) belief. But in many cases, it is important to be able to use quantifiers to express generality. For example, we might want to say not just that *this* belief of Kushim's is correct, but that *all* of Kushim's beliefs are correct. Or, to put the same point slightly differently, we might want to say:

(7) Everything Kushim believes is true.

Similarly, instead of attributing necessity specifically to Jupiter being a planet, we might want to say generally that necessity is not trivial, in the sense that there are examples of necessities. That is:

(8) Something is necessary.

As a final illustration, note that when I say in (1) ("If you invented the smiley, then I am the pope") that I am the pope, I am simply saying something which is patently false. I could equally have said that I am the king or queen of England, or made any other absurd claim. More generally, I might say that any absurdity is the case, or simply – and absurdly – that everything is the case:

(9) If you invented the smiley, then everything is the case.

Standard propositional languages do not provide any quantifiers, so there is no useful way of regimenting these three quantified claims in these languages. But it would take little to allow for such quantification: We would only have to allow ourselves to use sentential letters like p, q, and r as variables, and to bind them by a universal quantifier $\forall$ and an existential quantifier $\exists$, analogous to the familiar case of the quantifiers of first-order logic which bind individual variables. With such quantifiers, (7), (8), and (9) are straightforwardly formalized as follows:

(10) $\forall p(B_k p \rightarrow p)$
(11) $\exists p \Box p$
(12) $s \rightarrow \forall q\, q$

Logic is not just about formalizing statements, but also about capturing logical properties of, and relationships among, such statements. For example, (3) ("Necessarily, Jupiter is a planet") is an instance of the existential claim (8) ("Something is necessary"), and so the latter follows intuitively from the former. Thus, we would expect $\exists p \Box p$ to be counted as a logical consequence of $\Box r$. Using instances of classical propositional reasoning, this follows straightforwardly by a schematic principle of existential introduction, in particular the following instance:

(13) $\Box r \rightarrow \exists p \Box p$

Analogously, $s \rightarrow p$ can be obtained from $s \rightarrow \forall q\, q$ using the following instance of universal instantiation:

(14) $\forall q\, q \rightarrow p$

By classical propositional reasoning, $s \rightarrow \forall q\, q$ and $\forall q\, q \rightarrow p$ give us $s \rightarrow p$, as required.

Such quantifiers, binding variables which occupy the position of formulas, are often called *propositional quantifiers*, for example by Kripke (1959) and Fine (1970). This Element is about these quantifiers in the context of propositional languages, their logic, and their applications in philosophy. This introductory Section 1 explains first, in Section 1.2, why the job of propositional quantifiers cannot obviously be done by the more familiar quantifiers of first-order logic. Section 1.3 gives a partial explanation of why propositional quantifiers are rarely encountered: in the context of many logical systems, they are redundant in at least one of two different senses of redundancy. Section 1.4 explains why there is nevertheless a range of interesting settings in which propositional quantifiers are not redundant, and in which they are usefully studied. The main examples for this are propositional modal languages along the lines mentioned earlier. Consequently, the following Sections 2 and 3 of this Element are concerned with the resulting propositionally quantified modal logics. Section 1.5 provides an outlook on these sections and explains why they are structured according to different styles of set-theoretic model theories for such logics. A final Section 1.6 of the present section gives a brief overview of the historical development of propositional quantifiers in formal logic.

This Element presents a continuous narrative, but it is also possible to read the various sections selectively and out of order. Figure 1 presents a dependency diagram indicating which sections depend on which other sections. The label “PML” indicates that starting with Section 2, familiarity with the basics of propositional modal logic will be assumed. The central logical theory of propositionally quantified modal logics is developed in the section numbers highlighted in bold.

1.2 Why Propositional Quantifiers?

Before we get any further into the theory of propositional quantifiers, it is worth considering why one might want to use these quantifiers in the first place. In many contexts, quantificational claims can be captured straightforwardly using first-order logic, which is an extremely well-understood and very well-behaved

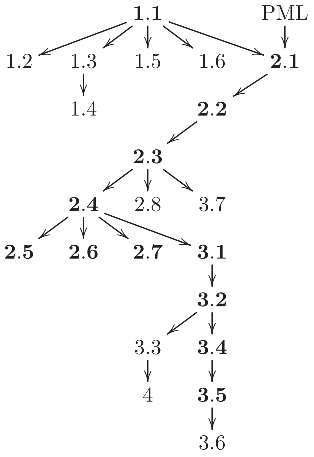

Figure 1 Dependency among sections

formal system. Can't we just use first-order logic to formalize the preceding examples?

In first-order logic, an atomic formula consists of an application of a predicate F to a finite number of arguments $t_1, \ldots, t_n$, forming the statement $Ft_1 \ldots t_n$. In the simplest case, the arguments are individual variables $x_1, \ldots, x_n$, which can be bound by first-order quantifiers $\forall$ and $\exists$. Thus, the claim that every echidna is happy can be formalized as follows, with the obvious interpretations of the predicate letters:

(15) $\forall x(Ex \rightarrow Hx)$

How might we use such first-order quantifiers to formalize, for example, (8) ("Something is necessary")? We cannot just replace a propositional variable by a first-order variable: $\Box$ is a sentential operator, and so $\Box$ can only be applied to formulas, and not to individual variables; thus the string $\exists x \Box x$ is ill-formed. There are two natural options to overcome this difficulty. First, we might use a necessity *predicate* N instead of a necessity *operator* $\Box$. With N, we might use the following formula of first-order logic:

(16) $\exists x Nx$

Second, we might introduce a truth predicate T. We can then attribute necessity to x by stating that x is necessarily true. With T, we might then use the following formula of first-order modal logic:

(17) $\exists x \Box Tx$

Consider first the option of using modal predicates. Using a predicate N instead of a sentential operator $\Box$ has a knock-on effect for the formalization of particular necessity claims. As noted, we want our formalization to capture that (8) ("Something is necessary") follows from (3) ("Necessarily, Jupiter is a planet"). But it is not clear how $\exists xNx$ could be seen to be a logical consequence of $\Box r$. The obvious response to this difficulty is to reconsider the formalization of (3), and use N instead of $\Box$. However, we can again not simply exchange N and $\Box$, since they take different types of expressions as arguments. In order to be able to apply N, Jupiter being a planet must be expressed by an individual term instead of a formula.

The problem would be solved if we could turn any formula, such as Fj, into a corresponding individual term. A natural idea is therefore to introduce a device which effects this transformation. So, let us consider an extension of the language of first-order logic in which for every formula φ, there is an individual term $[\varphi]$. For example, (3) ("Necessarily, Jupiter is a planet") can then be formalized in more detail as follows:

(18) $N[Fj]$

Now, $\exists xNx$ follows straightforwardly from $N[Fj]$ using existential introduction in first-order logic.

On the modal predicate approach, there are also cases which call out for a truth predicate. Consider (7) ("Everything Kushim believes is true"). As in the previous case, we might use a first-order quantifier instead of a propositional quantifier, and a predicate K instead of the sentential operator B_k to capture the notion of being believed by Kushim. With this, we can state that Kushim believes something using the formula $\exists xKx$. But it is not possible to formalize (7) along the lines of $\forall p(B_kp \rightarrow p)$; in particular, the string $\forall x(Kx \rightarrow x)$ is ill-formed, since x cannot take the position of a formula. The simplest way to address this deficiency is to introduce a truth predicate T. We can then propose to use the following formula:

(19) $\forall x(Kx \rightarrow Tx)$

The truth predicate T serves as something of an inverse of the propositional abstraction device $[\ldots]$ with which we obtain an individual term $[\varphi]$ from a formula φ: The first turns an individual term into a corresponding formula, and the second turns a formula into a corresponding individual term.

Thus, the modal predicate approach leads us to introducing a truth predicate. But, as we have seen, with a truth predicate we could also avoid introducing modal predicates and write, for example, $\Box Tx$ instead of Nx. It is also instructive to consider the inferential relationships between the various claims on this second approach. To capture that (8) ("Something is necessary") follows from (3) ("Necessarily, Jupiter is a planet"), we now would want $\exists x \Box Tx$ to follow from $\Box r$. Again, this means going from a sentential expression r to an individual variable x, which is naturally effected using the propositional abstraction device [. . .]. For example, one might appeal to the following schematic principle governing truth, where φ may be any formula:

(20) $\varphi \leftrightarrow T[\varphi]$

(This principle is closely related to the schematic biconditionals discussed by Tarski [1983 [1933]], but note that the latter deal with sentences rather than propositions.) Let φ be r, and consider the left-to-right direction: $r \rightarrow T[r]$. By standard axiomatic modal reasoning, using the rule of necessitation and the distributivity axiom for $\Box$, if $r \rightarrow T[r]$ is derivable, then so is $\Box r \rightarrow \Box T[r]$. From $\Box r$, we therefore obtain $\Box T[r]$, and so $\exists x \Box Tx$ by existential introduction.

Thus, although we can use first-order quantifiers instead of propositional quantifiers, both of the ways of doing so sketched here require further logical resources, in particular a truth predicate T and the propositional abstraction device [. . .]. One of them allows us to continue to use modal operators, as standard in modal logic; the other replaces them with modal predicates. It is worth noting that one might also endorse a hybrid approach which provides both modal operators and modal predicates; this could be justified by arguing that there is a philosophically important distinction between "necessarily" and "necessary" which gets conflated in standard uses of modal logic.

We have seen that there are ways of formalizing the quantificational examples of Section 1.1 in first-order logic. However, I hope to have illustrated that they come with certain complexities. In contrast, formalizations using propositional quantifiers are extremely simple. The complexities of first-order approaches may earn their keep by allowing for the formulation of theories which do better on other dimensions of theoretical virtue. This is not the place to try to settle these issues. For defenses of different versions of the first-order approach, see Halbach and Welch (2009) and Bealer (1998); for exchanges on the relative merits of the first-order and propositional approaches, see Anderson (1987) and Bealer (1994), as well as Menzel (2024) and Williamson (2024).

One straightforward reason for investigating propositional quantifiers is therefore their simplicity. Another has to do with the ontological commitments of different forms of quantification. When we regiment claims like (8) ("Something is necessary") using first-order quantifiers, it is natural to take these quantifiers to range over propositions. Existential claims like $\exists xNx$ and $\exists x \Box Tx$ therefore commit us to an ontology of propositions, namely to the existence of certain things – propositions – which are necessary. Nominalists, according to whom there are no propositions, will disagree with this claim. They might instead appeal to propositional quantifiers and argue that there is a way of interpreting the existential propositional quantification $\exists p \Box p$ which makes it an existential generalization of $\Box r$, without entailing the existence of propositions. On this view, propositional quantifiers should *not* be understood as ranging over propositions. How, then, should they be understood? According to one version of this view, propositional quantifiers should be thought of as a new, sui generis, form of quantification. In informal discussion, we might still paraphrase them using the available English constructions, but these paraphrases should – in the words of Frege (1892) – be taken with a grain of salt. Alternatively, we could introduce into English new constructions which better correspond to propositional quantifiers, as suggested by Prior (1971, section 3.4) and Grover (1972). In any case, on this use of logics with propositional quantifiers, there is a sense in which propositional quantifiers do not quantify over propositions. For more on these kinds of views, see Fritz and Jones (2024).

In order to avoid suggesting that propositional quantifiers quantify over propositions, such quantifiers are sometimes also called "sentential quantifiers"; see, for example, Künne (2003). Although this label brings out clearly that the relevant quantifiers bind variables which take *sentential* position, it also invites the suggestion that they are quantifiers ranging over sentences. In many cases, this would be a misunderstanding. There are *substitutional* readings of propositional quantifiers, and model-theoretic constructions in which the truth-conditions of propositional quantifiers are specified in terms of metatheoretic quantification over sentences; such approaches will be discussed in Section 3.7. But unless it is explicitly indicated that such a reading is intended, propositional (sentential) quantifiers should also not be taken as ranging over sentences.

1.3 Redundancy

I have argued that propositional quantifiers are natural and useful logical concepts, which cannot straightforwardly be replaced by more familiar forms of quantification in formal logic. Yet, propositional quantifiers are relatively underexplored. In this section, we observe one important reason for this: in many contexts, propositional quantifiers are redundant in at least one of two senses of redundancy.

To illustrate the first sense of redundancy, consider classical propositional logic. Assume that the formulation of this logic under consideration includes two logical sentential constants, $\top$ and $\bot$, with $\top$ and $\neg\bot$ being provable.

Since all sentential operators of classical propositional logic are truth-functional, any two materially equivalent formulas can be replaced, salva veritate, in any context. In this sense, classical propositional logic is *extensional*. Because of this, we can simulate propositional quantification straightforwardly: A universal quantification $\forall p\varphi$ is true just in case φ is true under any interpretation of p. By extensionality, since only the truth-value of φ matters, this is the case if and only if φ is true when p is replaced by $\top$ or $\bot$, for which we will write $\varphi[\top/p]$ and $\varphi[\bot/p]$, respectively. Instead of $\forall p\varphi$, we can therefore simply write $\varphi[\top/p] \wedge \varphi[\bot/p]$. Similarly, instead of $\exists p\varphi$, we can write $\varphi[\top/p] \vee \varphi[\bot/p]$.

The argument just sketched is essentially semantic, as it appeals to interpretations of the proposition letters. It is worth noting that a more careful version of the argument can be carried out deductively, using the principles of classical propositional logic and standard principles of elementary quantification for propositional quantifiers. The argument also extends to many other extensional logics, including classical first-order logic, second-order logic, and any extensions of these systems by generalized quantifiers: in all of them, adding propositional quantifiers is redundant along the lines sketched earlier. It will therefore be worth developing this argument a little more carefully and generally. We focus only on the universal propositional quantifier; the argument for the existential propositional quantifier is analogous and can also be derived from the universal case on the assumption that one quantifier is the dual of the other.

Let $\mathcal{L}^*$ be a logical language, the formulas of which are defined by a standard recursion, and let $\vdash^*$ be a proof system for $\mathcal{L}^*$. We write $\vdash^* \varphi$ for a formula φ of $\mathcal{L}^*$ being provable in $\vdash^*$. We assume the following:

- We have defined the notion of an occurrence of a propositional variable p being *free* in a formula φ, and the notion of a formula ψ being *free for* a propositional variable p in a formula φ. (These are standard notions of logical systems; see Section 2.1 for definitions in the case of propositionally quantified modal logic.) If the latter condition is satisfied, we write $\varphi[\psi/p]$ for the result of replacing every free occurrence of p in φ by ψ.
- $\mathcal{L}^*$ is closed under the Boolean connectives, including $\top$ and $\bot$, as well as the universal propositional quantifier $\forall p$, so that $\forall p\varphi$ is a formula whenever φ is a formula.
- The formulas provable in $\vdash^*$ include all classical tautologies and are closed under modus ponens (MP), the rule that if $\vdash^* \varphi$ and $\vdash^* \varphi \rightarrow \psi$, then $\vdash^* \psi$.
- $\vdash^*$ includes the principles of *universal instantiation* (UI) and *universal generalization* (UG) for propositional quantifiers. So, if ψ is free for p in φ, then

$$\vdash^* \forall p\varphi \to \varphi[\psi/p],$$

and if p is not free in φ, then

$$\vdash^* \varphi \to \psi \text{ only if } \vdash^* \varphi \to \forall p\psi.$$

- $\mathcal{L}^*$ is *extensional* according to $\vdash^*$, in the sense that whenever ψ and χ are free for p in φ, then:

$$\vdash^* (\psi \leftrightarrow \chi) \to (\varphi[\psi/p] \leftrightarrow \varphi[\chi/p])$$

It follows from these assumptions that the formulas provable in $\vdash^*$ are closed under uniform substitution, in the sense that $\varphi[\psi/p]$ is provable whenever φ is provable and ψ is free for p in φ; the argument is analogous to the proof of Proposition 2.2.2.

The elimination of propositional quantifiers can be stated more rigorously by defining a mapping $\cdot^*$ from $\mathcal{L}^*$ to the sublanguage of $\mathcal{L}^*$ in which no universal propositional quantifiers occur. This mapping is defined recursively, with only one nontrivial clause:

$$(\forall p\varphi)^* := \varphi^*[\top/p] \wedge \varphi^*[\bot/p]$$

To say that the mapping is recursive and all other cases are trivial is just to say that $\top^* := \top$, $(\neg\varphi)^* := \neg(\varphi^*)$, and so on.

We can now prove that this elimination succeeds by showing that it maps any formula to a provably equivalent formula which does not contain any universal propositional quantifiers:

Claim For every formula $\varphi \in \mathcal{L}^*$, $\vdash^* \varphi \leftrightarrow \varphi^*$.

Argument The argument is by induction on the complexity of φ. By the construction of the mapping $\cdot^*$, it suffices to consider just the case of universal propositional quantifiers, and show that $\vdash^* \forall p\varphi \leftrightarrow (\forall p\varphi)^*$ on the assumption that $\vdash^* \varphi \leftrightarrow \varphi^*$. We consider the two directions of the biconditional separately.

For the left-to-right direction, note first that by UI, tautologies, and MP,

$$\vdash^* \forall p\varphi \to \varphi[\top/p] \wedge \varphi[\bot/p].$$

Second, by induction hypothesis (IH), $\vdash^* \varphi \to \varphi^*$. So, for $+$ being $\top$ or $\bot$, it follows by uniform substitution that

$$\vdash^* \varphi[+/p] \to \varphi^*[+/p].$$

Using tautologies and MP, we obtain, from these two observations,

$$\vdash^* \forall p\varphi \to \varphi^*[\top/p] \wedge \varphi^*[\bot/p].$$

This is $\vdash^* \forall p\varphi \to (\forall p\varphi)^*$, as required.

For the right-to-left direction, note first the following two instances of extensionality:

$$\vdash^* (p \leftrightarrow \top) \rightarrow (\varphi^* \leftrightarrow \varphi^*[\top/p])$$
$$\vdash^* (p \leftrightarrow \bot) \rightarrow (\varphi^* \leftrightarrow \varphi^*[\bot/p])$$

By tautologies and MP, we conclude:

$$\vdash^* \varphi^*[\top/p] \wedge \varphi^*[\bot/p] \rightarrow \varphi^*$$

By IH, $\vdash^* \varphi^* \rightarrow \varphi$, and so:

$$\vdash^* \varphi^*[\top/p] \wedge \varphi^*[\bot/p] \rightarrow \varphi$$

Since p is not free in the antecedent of this conditional, we can apply UG:

$$\vdash^* \varphi^*[\top/p] \wedge \varphi^*[\bot/p] \rightarrow \forall p\varphi,$$

which is $\vdash^* (\forall p\varphi)^* \rightarrow \forall p\varphi$, as required. This concludes the argument.

It is routine to show that many standard logical systems, including classical propositional logic, first-order logic, and second-order logic, all satisfy the assumptions of this result when propositional quantifiers are added to the syntax and the deductive principles UI and UG are added to a standard axiomatic proof system. The crucial condition of extensionality can be shown by induction on the complexity of formulas. In all these logics, propositional quantifiers are therefore redundant in the sense that they can be eliminated along the lines developed here.

The eliminability of propositional quantifiers shows that for many purposes, it is pointless to add propositional quantifiers. However, not every property one may be interested in is preserved under the elimination. Most importantly, the elimination of propositional quantifiers leads to an exponential increase in the size of formulas. Consequently, the elimination of propositional quantifiers may not preserve matters of computational complexity of the logic in question. This can be illustrated by the basic case of classical propositional logic. The problem of determining whether a formula of classical propositional logic is satisfiable (i.e., is true under some assignment of truth values to the proposition letters) is NP-complete, whereas the corresponding satisfiability problem for classical propositional logic with propositional quantifiers is PSPACE-complete. (For further details, including definitions of these complexity classes, see Blackburn et al. [2001, pp. 514–516]. That PSPACE-completeness comes apart from NP-completeness is conjectured, but has not been proven.) Classical propositional logic with propositional quantifiers is also known as QBF ("quantified Boolean formulas"); for applications in artificial intelligence and