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978-1-009-51767-6 — Definitions and Mathematical Knowledge
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Excerpt
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1 Introduction

Since antiquity, definitions have played a crucial role in the organization of our rational inquiries. Many of Plato's dialogues, to mention a familiar paradigm, can be seen as excruciating quests for definitions. The more definite the language and subject matter of our inquiries are, the more rigorous the treatment of definitions will be. Mathematics is thus a privileged arena. Given the exactness that mathematical contexts require, the study of definitions reached a mature stage once formal tools became sophisticated enough. Hence, most of the following discussion is indebted to the advancements that have occurred since the mid nineteenth century in mathematics, logic, and philosophy. The reliance on formal tools should not be overestimated though, for many theoretical features of definitions bear relevance to mathematical practice even before and beyond formalization.

As Euclid's *Elements* and Aristotle's *Posterior Analytics* emphasized, definitions are essential to any systematic inquiry. They are central to any project of conceptual analysis, especially to foundational projects. If these are seen as aiming at the attainment and organization of knowledge, definitions take on a genuine epistemological function. This Element focuses on the ways in which definitions may constitute, provide, or otherwise lead to mathematical knowledge.

What follows is thought of as an initial guide to a vast debate, which has nevertheless rarely been given a self-standing treatment (notable exceptions are Robinson 1950; Suppes 1957, chapter 8; Dubislav 1981; Belnap 1993; Antonelli 1998; Gupta and Mackereth 2023). We first rehearse the role of definitions in foundational projects (Section 2). We then discuss three major kinds of definitions: explicit definitions (Section 3), implicit definitions (Section 4), and implicit definitions of primitive terms (Section 5), the latter being divided into axiomatic (Section 5.1) and abstractive (Section 5.2). After pausing on the notions of elucidation and explication (Section 6), we eventually survey (Section 7) a variety of epistemological issues concerning definitions. We'll look for a balance among historical context, formal tools, and philosophical investigations, assuming some background but with the inexperienced reader in mind. Most mathematical examples will be confined, for analogous reasons, to geometry, arithmetic, and analysis. A consistent part of our discussion will concern authors who pioneered the foundations of mathematics (prominently Dedekind, Frege, Hilbert, Peano, Russell, and Carnap), and proponents of major contemporary views. Logicist and structuralist views, and a comparison between the two, will be given special attention, due to their role in foundational debates and their reliance on two major sorts of definitions for mathematical primitives.

2 Definitions and Foundations

2.1 Foundations

A vivid outline of a foundational project for mathematics is offered by Quine (1969), 69–70:

> Studies in the foundations of mathematics divide symmetrically into two sorts, conceptual and doctrinal. The conceptual studies are concerned with meaning, the doctrinal with truth. The conceptual studies are concerned with clarifying concepts by defining them, some in terms of others. The doctrinal studies are concerned with establishing laws by proving them, some on the basis of others. Ideally the more obscure concepts would be defined in terms of the clearer ones so as to maximize clarity, and the less obvious laws would be proved from the more obvious ones so as to maximize certainty. Ideally the definitions would generate all the concepts from clear and distinct ideas, and the proofs would generate all the theorems from self-evident truths.

Deductive *proofs* transfer (truth, justification, and) certainty from basic truths to theorems, while *definitions* transfer (meaning and) clarity from basic to derivative concepts. This marks a distinctively epistemological enterprise, echoing broader projects in the foundation of knowledge – Descartes' on the rationalist side (see e.g. the *Discourse on Method*, Descartes 1637), or Hume's on the empiricist side (see e.g. Hume 1739–40). More generally, it shares elements with a traditional model for the systematization of a deductive science (de Jong and Betti 2010), starting from Aristotle's *Posterior Analytics*. But what exactly is an epistemic foundation meant to provide?

On a strong reading, foundations deliver knowledge in the subject matter of a target domain D that was previously precluded. A notable drawback is that before foundations no one could genuinely be credited with knowledge of D. If we follow a (debatable, but still traditional) tripartite definition of (propositional) knowledge as justified true belief, then since foundations secure true beliefs, of which some, at least, were antecedently possessed, such beliefs can fail to count as knowledge only because they were not justified either. Only a foundation establishes that certain p's in D are true, and provides reasons to justifiedly believe in any (true) p in D. Such strong reading may have been underlying projects like Descartes', where hyperbolic doubt challenges both truth and justification for all beliefs in order to attain those we cannot possibly doubt; in our present case, the implication that mathematical beliefs could not justifiedly be held true until the nineteenth century is obviously unpalatable.

On a more indulgent reading, foundations have an *architectural* purpose: their aim is not (just) to question whether target statements are true, or whether we have any good reasons to believe that they are, but to establish conclusively *why* they are true, to determine the ultimate grounds their truth depends on. This approach often looks back at Euclid's *Elements* as a paradigm. Descartes himself (*Discourse*, II, 19) was moved by similar thoughts: "Those long chains of utterly simple and easy reasonings that geometers commonly use to arrive at their most difficult demonstrations had given me occasion to imagine that all the things that can fall within human knowledge follow from one another in the same way […]."

At the beginning of the modern debate on the foundation of mathematics, Frege (1884) (along with Bolzano, Dedekind, and others, with due differences) endorsed similar views:

> The aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another. After we have convinced ourselves that a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely? (§2)

Rather than leading from lack to possession of knowledge, then, foundations may be concerned with different kinds of justification. They would replace a weaker, defeasible, possibly even *a posteriori* and inductive justification based on successful applications, with the incontrovertible justification provided by an explanation of the deductive relations connecting basic principles and theorems. This architectural approach can then be accompanied, or even motivated, by purely mathematical concerns as to how a mathematical theory is to be best systematized.

On the epistemological side, foundations require establishing how basic principles are themselves justified, or otherwise warranted, in a noninferential way. On pain of regress, as Aristotle made clear, inferential justifications must come to an end. Definitions must come to an end too. Basic terms cannot be defined from more elementary ones, and still their meaning must be available somehow. Foundations seem bound to feature both *unprovable principles* and *indefinable primitives*. Both issues are especially pressing in mathematics, whose truths and objects, on most conceptions, are inaccessible empirically or extratheoretically.

Within a foundational project, definitions can provide, or sustain, different varieties of epistemic achievements. Surely they afford an *understanding* of the meaning of the linguistic items that are being defined, or the concepts they express, required to master them competently. They can sustain *propositional knowledge* (knowledge that something is the case) with respect to both the statements which are used to lay down the definition itself, and the theorems that can be derived thanks to its introduction. They may even lead to *objectual knowledge*, namely knowledge of the objects (if any, and of any variety) of a mathematical theory. If at least some of the basic concepts F of a mathematical theory M are concepts under which objects supposedly fall, by determining when an object x is to fall under F a definition will show how to individuate objects of kind F, and the truth of M will entail countenancing their existence. In this respect, a foundation is also an ontological, and possibly metaphysical, enterprise. Its aim is to determine, according to a given theory, what kinds of objects there are and what kinds of objects they are (what their nature, or essence, is). It can also determine what kinds of objects there aren't, if they can be reduced to, or identified with, objects of other kinds. Accordingly, although our main focus will be on epistemology, matters in both semantics and ontology will also play a relevant role.

Foundationalism can come in many varieties (Shapiro 2004), some of which emphasize the theoretical and scientific significance of systematizing, unifying, and connecting mathematical theories over traditional epistemological concerns with evidence, knowledge, and justification. Moreover, even granting that foundations are possible at all, foundationalism does not exhaust the philosophy of mathematics, nor, therefore, the epistemology of definitions. Various issues concerning definitions arise when we look at phenomena like mathematical explanation or understanding, when we consider why mathematicians redefine already established notions, or when we discuss whether one definition is more natural than another; these issues are partly independent of foundational projects, may be influenced by cognitive, sociological, and pragmatic factors, and are elicited by the study of past and current mathematical practice (see e.g. Tappenden 2008; Frans, Coumans, and de Regt 2022; Coumans 2024). Some of these will cursorily surface in what follows, although this more nuanced investigation of definitions will be kept in the background while we attempt to outline and systematize a debate which is closer to traditional foundational concerns.

But what are definitions, and what guides their formulation? Definitions are statements expressing, or establishing, a relation between (the meaning of) some linguistic items and (the meaning of) other linguistic items. They establish that the meaning of a hitherto undefined symbol or expression of a given grammatical type, the *definiendum*, is to be determined on the basis of the meaning of a combination of one or more previously known symbols or expressions, the *definiens*. Relevant grammatical types in natural language are singular terms, including proper names, and predicates, including relational expressions. Within a formal system, relevant syntactic types are constants, function symbols (including term-forming operators), and predicate and relational symbols. The way in which such a determination of meaning is effected (e.g. whether there is some sort of semantic equivalence between *definiens* and *definiendum*, whether both have to be of the same syntactic or grammatical type, and so on) varies, as we will see, according to the specific kind of definition considered.

Formally, the effect of the definition is to assign the defined expression a suitable syntactic and inferential behavior. Semantically, to a very first approximation, it is to provide the *definiendum* with a set of one or more (individually necessary and jointly sufficient) defining conditions determining its semantic interpretation (usually, the individual they refer to for constants and singular terms, the function they determine for functional symbols, or the set of individuals or n-tuples of such individuals they denote for predicate and relation symbols). On some conceptions, definitions also target extralinguistic items by individuating the essence, nature, or constitutive features of some object or property.

Some definitions can be seen as arbitrary stipulations that a certain expression is to be given a certain meaning. In the most interesting cases, however, definitions are guided by pretheoretical patterns of use of informal notions that they aim at capturing (wholly or partially) and systematizing. This raises the question of how to establish whether a definition is successful (and what it is, in general, for a definition to be successful). Usually, this involves a process of *reflective equilibrium* (Goodman 1954/1983) between evidence and theory: we systematize patterns of use through defining conditions, then check whether the definition thus obtained is too strict (leaves out cases we would like it to cover) or too loose (it applies to cases we wouldn't want it to apply to), and then go back to adjusting the defining conditions so as to make the definition more adequate and precise. The details and significance of this procedure depend, as we will see, on different conceptions of conceptual analysis.

2.2 Names and Things

Traditionally, a distinction is drawn between *nominal* and *real* definitions, that is, definitions *of names* (*quid nominis*) versus definitions *of things* (*quid rei*). The distinction is examined in Aristotle's remarks on definitions (Deslauriers 2007), especially in *Posterior Analytics* (CW, I), and has been later related to definitions in Euclid's *Elements* (e.g. by Saccheri; see Heath's commentary to Euclid 1926, I). To a first approximation, nominal definitions target linguistic items, that is, they provide linguistic expressions with meaning (hence, the concepts they express with content); real definitions target the objects themselves and capture their essential properties.

On one reading of Aristotle, real definitions differ from nominal ones because they reveal *why* an object is (what is the cause, or *aitia*, of their existence) and give "a demonstration of the essence" (*Posterior Analytics*, II, 94a) of a thing. They provide a metaphysical explanation. On a weaker reading, real definitions are nominal definitions together with the assumption that what is being defined exists: so, while nominal definitions just answer the question *what* a thing is (*to esti*) with no commitment to its existence, other definitions also presuppose an answer to the question *whether* a thing is (*ei esti*) and claim that it is (*oti esti*). As such, they are especially suited for primitive notions, the existence of whose referents cannot be proved within the theory and must hence be presupposed. Nominal definitions simpliciter pertain to derivative notions, the existence of whose objects has to be proved from general principles common to all sciences (common axioms, *koinà axiomata*) and from the postulates and primitive notions of the relevant theory:

> Now the things peculiar to the science, the existence of which must be assumed, are the things with reference to which the science investigates the essential attributes, e.g. arithmetic with reference to units, and geometry with reference to points and lines. With these things it is assumed that they exist and that they are of such and such a nature. . . . But, with regard to their essential properties, what is assumed is only the meaning of each term employed: thus arithmetic assumes the answer to the question what is (meant by) 'odd' or 'even', 'a square' or 'a cube,' and geometry to the question what is (meant by) 'the irrational' or 'deflection' or (the so-called) 'verging' (to a point); but that there are such things is proved by means of the common principles and of what has already been demonstrated. (*Posterior Analytics* I, 10, 76b3–76b9)

For instance, geometers must assume that there are points, but they define 'triangle' nominally, leaving to demonstrations or geometrical constructions the task of proving that triangles exist. As regards primitives, or "immediates," Aristotle further writes:

> Of some things there is something else that is their explanation, of others there is not. Hence it is clear that in some cases what a thing is is immediate and a principle; and here one must suppose, or make apparent in some other way, both that they are and what they are (which the arithmetician does; for he supposes both what the unit is and that it is); but in those cases which have a middle term [in a syllogism] and for which something else is explanatory of their substance, one can, as we said, make them clear through a demonstration, but not by demonstrating what they are. (*Posterior Analytics* II, 9, 93b22–93b28)

Definitions of immediates are counted (with common axioms and postulates) among the *archai*, the first indemonstrable principles of a deductive science. What distinguishes immediates from other notions is that an explanation of why they are – their *aitia* – is not "something else": they cannot be explained by reduction to simpler items. Here nominal definitions come to an end, and the content of immediates must be made "apparent in some other way," for they provide the very subject matter of the theory itself, its *genus* (*genos*).

The distinction between nominal and real definitions has been preserved and discussed in the history of philosophy (see e.g. Locke 1690, III, vi) and has elicited varied responses. Within the analytic tradition, due to its inherent skepticism toward traditional metaphysics, real definitions have been dismissed. To wit, in discussing Aristotle on definitions, Russell (1945, 223–225), contends that the notion of essence, like that of substance, is "a hopelessly muddle-headed notion . . . a metaphysical mistake, due to transference to the world-structure of the structure of sentences composed of a subject and a predicate." Other earlier reactions were friendlier. Mill (1843, I, VIII, 7), for instance, acknowledged that "definitions, though of names only, must be grounded on knowledge of the corresponding things […] How to define a name […] may involve considerations going deep into the nature of the things which are denoted by the name. Such, for instance, are the inquiries which form the subjects of the most important of Plato's Dialogues."

Aristotle's concern for immediates as providing the *genus* of a theory is indeed due to his reaction to Plato's account of definitions. Many Platonic dialogues are driven by questions of the "What is *X*?" form. These are best seen as requests for real definitions of the essence of *X* (love, knowledge, etc.), rather than nominal definitions of *X*-terms. In the *Sophist* (see also *Phaedrus*, 265d–266b), Socrates applies the so-called *method of collection and division*: to find a correct definition of *X*, the inquirer should first collect different examples of what we take to be *X*, consider them as falling under some broader kind, and then proceed, by a dichotomic process, by dividing the largest kind into two, locating our target *X* into one smaller kind, to be then further divided, and so on. The essence of the species (*eidos*) *X* is then individuated by giving its *genus* and its *differentia* (*diaphora*), which distinguishes it from other species under the same *genus*. This decompositional analysis of concepts distinguishes proper definitions from lists of cases. When asked "What is knowledge?," the young mathematician Theaetetus (*Theaetetus*, 146c–d) initially offers a list of familiar examples: geometry, cobblery, carpentry, and "the skills that belong to other craftsmen." Socrates is adamant that this is not what he is looking for. He asks for what is common to all these examples and requires a general *criterion* for establishing whether some yet unencountered item is or isn't a case of knowledge. In contemporary terminology, this requires a definition of a concept *X* to provide a set of *individually necessary* and *jointly sufficient* conditions for the application for *X*, and, if it is a concept of objects, to determine when two objects falling under the concept are distinct (conditions of identity), and whether an object falls under the concept or not (conditions of application).

Aristotle does not reject definitions of this kind, but doubts we can rely on them only. One objection is that the process of division by *genera* assumes what it wants to prove – that is, assumes the existence of the objects that we subsume progressively into broader *genera*. On the contrary, many definitions can take the form of conclusions of syllogistic reasonings: since definitions are a "demonstration of essences," their statement can occur as the conclusion of arguments (*Posterior Analytics*, II.3–10). This other form of definition is for Aristotle much more suited to a general method of a demonstrative science, while "division by genera is a small part" of such a method.

Although mathematical definitions are most of the time treated as nominal, they are still often (implicitly or explicitly) treated as real definitions and may be conceived as ways to capture the nature of mathematical objects or to determine the conditions for their existence (see Section 7.4.2).

2.3 The Euclidean Paradigm

Both Plato's and Aristotle's views on definitions were plausibly influenced by Greek geometry, and themselves have influenced the reception of its later presentation in Euclid's *Elements*. Since the latter have long been a paradigm of foundational theories, recalling even a much simplified outline will help (for more, see Euclid 1926, introduction; Mueller 1981).

Book I of the *Elements* (in Heidberg's edition) contains twenty-three definitions (ὅροι), five postulates (αἰτήματα), and five common notions (κοινάι ἔννοιαι). A sample of the first seven definitions (with emphasis added on the term being introduced) is:

Euclid's Definitions (sample)

D1 A **point** is that which has no part.
D2 A **line** is breadthless length.
D3 The extremities of a line are points.
D4 A **straight line** is a line which lies evenly with the points on itself.
D5 A **surface** is that which has length and breadth only.
D6 The extremities of a surface are lines.
D7 A **plane surface** is a surface which lies evenly with the straight lines on itself.

In Aristotle's terms, we find both immediates and nominal definitions of derivative notions. 'Point,' 'line,' and 'surface' are among the former. They identify the subject matter (*genus*) of the theory, whose existence – according to Aristotle – must be posited, and whose explanation does not depend on "something else." Other notions are derived from them. For instance, D7 defines 'plane surface' through the already available notions of surface and straight line, the latter being itself introduced in D4 in terms of line and point. Remaining expressions – 'part,' 'breadth,' 'extremities,' 'inclination,' 'meet,' 'lies evenly,' 'containing', and so on – may be seen as part of an antecedently shared language (note that extremities could be taken as primitives and used to define other notions in D3 and D6, although this would violate Aristotle's concern not to "explain the prior by the posterior"; Aristotle CW, *Topics*, I, 4, 141b.15 ff.). Following this taxonomy, statements introducing primitives (D1, D2, etc.) would not properly count as definitions, but rather illustrations or elucidations (Section 6.1) of notions whose basic grasp is antecedently and pretheoretically guaranteed (for instance by spatial intuition). So much so that they are actually never used in proofs in the *Elements*, a feature that takes them apart from other proper definitions (and raises both exegetical issues on the composition of Euclid's work – Russo 1998 – and conceptual questions as to their theoretical role as purported definitions – see Section 3.3).

The five postulates are:

Euclid's Postulates

P1 To draw a straight line from any point to any point.

P2 To produce a finite straight line continuously in a straight line.

P3 To describe a circle with any centre and distance.

P4 That all right angles are equal to one another.

P5 That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Postulates are not proved from anything else, and are supposed to use only primitive notions and notions already defined through them (e.g. 'straight line' is defined in D4 via D2 and D1). Differently from modern presentations of mathematical axioms, some of Euclid's postulates (e.g. P1–P3) are not properly assertions (declarative descriptions of geometrical facts supposed to hold), but rather prescriptions or instructions on how certain figures can be constructed (through ruler and compass). Also, we now call 'axioms' the principles of a theory, but ancient usage displays a subtle variety of uses for the term (Euclid 1926, Introduction, IX, §3).

Definitions and Postulates are supplemented with common notions:

Euclid's Common Notions

- C1 Things which are equal to the same thing are also equal to one another.
- C2 If equals be added to equals, the wholes are equal.
- C3 If equals be subtracted from equals, the remainders are equal.
- C4 Things which coincide with one another are equal to one another.
- C5 The whole is greater than the part.

These are general principles concerning quantity and magnitude, and offer the theoretical scaffolding for a derivation of geometrical theorems. Notice that proofs in the *Elements* heavily rely on actually constructed diagrams of geometrical figures. A proper separation between their propositional and visual aspects may be hard to draw and diagrams may contribute to the justification of theorems as ingredients of their derivation (Giaquinto 2007; Manders 2008).

Several other definitions of great mathematical value are found in the later books. Worth mentioning are at least those concerning ratios of magnitudes in Book V, systematizing Eudoxus' theory of proportions, which will underlie later characterizations of real numbers; and those, in Books VII–IX, illustrating the notion of unit ("that by virtue of which each of the things that exist is called one") and defining number as "a multitude composed of units" (also introducing other notions like multiple, even, odd, prime, etc.). Possibly because arithmetic, on this conception, requires no proper construction procedures (apart from the addition of one unit to other units), no proper arithmetical postulates are provided, so no axiomatic treatment of arithmetic is advanced. This was to change only in the nineteenth century.

This brief outline not only shows that Aristotle's taxonomy of definitions applies (being itself inspired by geometrical practice) to the definitions in the *Elements*. It also shows – once significant idealizations or simplifications are conceded – why the *Elements* have been the paradigm model of foundations, where certain truths lead to theorems via proof, and clear notions lead to derivative ones via definition. Whether such a paradigm is still adequate for modern conceptions of mathematical knowledge is something we will touch upon and that can variously be disputed (Paseau and Wrigley 2024). Finally, it emphasizes the irreducible role of postulates – which must be noninferentially justified – and primitives – whose meaning cannot be given in simpler terms.