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MATHEMATICAL PLURALISM

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Abstract: Mathematical pluralism is the view that there is an irreducible plurality of pure mathematical structures, each with its own internal logics, and that qua pure mathematical structures they are all equally legitimate. Mathematical pluralism is a relatively new position on the philosophical landscape. This Element provides an introduction to the position.

Keywords: mathematical pluralism, mathematical foundationalism, mathematical objects, applied mathematics, non-classical logic

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Foreword

Since I came to philosophy from mathematics, my first philosophical love was the philosophy of mathematics. I have acquired many other philosophical interests since then; but I have never lost a philosophical interest, and every new interest makes the old interests richer. So it is that I have regularly returned to the philosophy of mathematics, seeing old issues in new lights. In fact, over the years, I think my views on the philosophy of mathematics have changed more than those concerning any other area of philosophy. I was never sympathetic to platonism;[1] but the landscape of the philosophy of mathematics (to say nothing of the rest of philosophy) has changed substantially in the last 50 years, and (it seems to me) there are now much better ways of framing an anti-platonist view.

In particular, over the last 10 years or so I have become sympathetic to mathematical pluralism. So when Penny Rush and Stewart Shapiro approached me to write a short Element on the topic in their Philosophy of Mathematics *Cambridge Elements* Series, I was very happy to accept. This provided a welcome opportunity to attempt to weld a number of things I have written on the topic in the last few years into a (hopefully!) coherent whole. These are referenced in what follows.[2] Section 1 largely reproduces Priest (2019a), and Section 3 largely reproduces Priest (202+a). I am grateful to the editors and publishers of those pieces for permission to reuse the material.

Many thanks go to a number of friends who, in commenting on earlier drafts of the manuscript, parts thereof, or in conversation, gave me valuable thoughts and criticisms. These include Justin Clarke-Doane, Hartry Field, Will Nava, Lavinia Picollo, Andrew Tedder, and Elia Zardini. Thanks go to Joel Hamkins for technical help on set theory. I taught a course on the philosophy of mathematics at the CUNY Graduate Center in the Spring semester, 2022, where my students read (amongst other things) a draft manuscript and gave me valuable comments and criticisms. Many thanks go to them too. A special thanks goes to Stewart Shapiro. Stewart read what I expected to be essentially the final draft of the manuscript, and his perceptive comments and suggestions led to the current improved version.

1 As can be seen from Priest (1973).
2 My first foray into the area was Priest (2013a).

Preface

One may hold there to be a unity to mathematics; that is, that there is one overarching framework into which all of (true) mathematics fits. I doubt that one finds this view in the history of mathematics before the twentieth century. At least before Descartes, there was a pretty rigid distinction between arithmetic (the investigations of numbers) and geometry (the investigation of spatial relations). And after the invention/discovery of non-Euclidean geometries, there was, I think, no sense that they could all be subsumed under a unifying framework before Riemann.

In the twentieth century, the view that there was such a unity to mathematics did, however, become orthodox. The unifying framework was that of Zermelo–Fraenkel set theory – usually, with the Axiom of Choice – *ZFC*. That *ZFC* played this role emerged from studies in the foundations of mathematics in the late nineteenth and early twentieth centuries. As one highly respected philosopher of mathematics, Penelope Maddy, describes this moment in the history of mathematics/philosophy:[3]

> The view of set theory as a foundation for mathematics emerged early in the thinking of the originators of the theory and is now a pillar of contemporary orthodoxy. As such, it is enshrined in the opening pages of most recent textbooks; to take a few illustrative examples:
>
> *All branches of mathematics are developed, consciously or unconsciously, in set theory. (Levy (1979), 3)*
>
> *Set theory is the foundation of mathematics. All mathematical concepts are defined in terms of the primitive notions of set and membership . . . From [the] axioms, all known mathematics may be derived. (Kunen (1980), xi)*
>
> *[M]athematical objects (such as numbers and differentiable functions) can be defined to be certain sets. And the theorems of mathematics (such as the fundamental theorem of calculus) then can be viewed as statements about sets. Furthermore, these theorems will be provable from our axioms. Hence, our axioms provide a sufficient collection of assumptions for the development of the whole of mathematics – a remarkable fact. (Enderton (1977), 10–11)*
>
> From its Cantorian beginnings through its modern flowerings, set theory has also raised problems of its own, like any other branch of mathematics, but its larger, foundational role has been and remains conspicuous and distinctive.

What it means to say that a theory is foundational for mathematics is no straightforward matter, and different philosophers have held somewhat

3 Maddy (1997), p. 22.

different views on the matter. Maddy discusses various possible interpretations of the view: ontological, epistemological, and methodological. But the details here need not concern us.[4] However one understands what, exactly, it amounts to, *ZFC* is taken to be a single framework into which all of mathematics, in some sense, fits. As Maddy herself summarises matters:[5]

> Finally, perhaps most fundamentally, *this single, unified arena for mathematics* provides a court of final appeal for questions of mathematical existence and proof: if you want to know if there is a mathematical object of a certain sort, you ask (ultimately) if there is a set theoretic surrogate of that sort; if you want to know if a given statement is provable or disprovable, you mean (ultimately), from the axioms of the theory of sets.

And again:[6]

> set theoretic foundations . . . *play a strong unifying role*: vague structures are made more precise, old theorems are given new proofs and *unified with other theorems* that previously seemed quite distinct, similar hypotheses are traced at the basis of disparate mathematical fields, existence questions are given explicit meaning, unprovable conjectures can be identified, new hypotheses can settle old open questions, and so on. That set theory plays this role is central to modern mathematics, that it is able to play this role is perhaps the most remarkable outcome of the search for foundations.

That there is a unity to mathematics became, as she says, the orthodox view amongst philosophers of mathematics in the twentieth century. Arguably, it still is.[7]

However, the view is now starting to give way to one according to which there is no such unity. Mathematics is irreducibly a plurality. There is no grand narrative into which it can all be fitted. Indeed, it is investigations into the area of the foundations of mathematics which have themselves brought the view to breaking point.

One may call the emerging view, naturally enough, *mathematical pluralism*, and the point of this Element is to explain and examine the view. It is no impartial guide, however. It also endorses and argues for the view. It is none

4 For one clear discussion of the matter, see Shapiro (2004).
5 Maddy (1997), p. 26. My italics in this and the next quotation.
6 Maddy (1997), p. 34 f.
7 In a later publication Maddy reiterates her position, though she restricts it, without much explanation, to 'classical' mathematics (Maddy (2007), p. 354). What she means by 'classical mathematics' is not entirely clear; but if it means mathematics based on classical logic, even this more restricted claim runs aground on classical mathematical theories that do not fit into *ZFC*, as we will note in 2.2.

the worse, I think, for that. New views need advocates to make their full force felt. Orthodoxy will never lack its conservative defenders.

In Section 1, we will look at the evolution of studies in the foundations of mathematics in the twentieth century, and see how mathematical pluralism arose naturally out of these. Then in Section 2, we will have a closer look at mathematical pluralism itself, some of its features, and some possible objections.

The mathematics I have been talking about and which is discussed in the first two sections is pure mathematics. But mathematics also encompasses applied mathematics. What is one to make of this on a pluralist view? Section 3 investigates. Unsurprisingly, the picture which emerges is different from that which normally goes with set-theoretic foundationalism.

As hardly needs to be said, the nature of mathematics is deeply entangled with the nature of logic. Here is not the place to discuss all matters involved in the bearing of mathematical pluralism on that topic. However, the last section in the Element, Section 4, discusses what I take to be some of the most important issues.

Of course, in an Element of this length it is inevitable that a number of important issues receive no more than a cursory discussion. However, by the end of our short journey through the terrain of mathematical pluralism, you, the reader, will, I hope, have a decent understanding of the view and its features. What to make of it is, as ever in philosophy, up to you to decide.